Virasoro Frames and their Stabilizers for the E_8 Lattice type Vertex Operator Algebra

Robert L. Griess, Jr.* and Gerald Höhn†

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Abstract

The concept of a framed vertex operator algebra (FVOA) is new (cf. [DGH]). This article contributes to this theory with a full analysis of all Virasoro frame stabilizers in V, the important example of the E_8 level 1 affine Kac-Moody VOA, which is isomorphic to the lattice VOA for the root lattice of $E_8(\mathbb{C})$. We analyze the frame stabilizers, both as abstract groups and as subgroups of $Aut(V) \cong E_8(\mathbb{C})$. Each frame stabilizer is a finite group, contained in the normalizer of a $2B$-pure elementary abelian 2-group in $Aut(V)$, but is not usually a maximal finite subgroup of this normalizer. In particular, we prove that there are exactly five orbits for the action of $Aut(V)$ on the set of Virasoro frames, thus settling an open question about V in Section 5 of [DGH]. The results about the group structure of the frame stabilizers can be stated purely in terms of modular braided tensor categories, so this article contributes also to this theory.

There are two main viewpoints in our analysis. The first is the theory of codes, lattices, markings and the resulting groups of automorphisms. The second is the theory of finite subgroups of Lie groups. We expect our methods to be applicable to the study of other FVOAs and their frame stabilizers. Appendices present aspects of the theory of automorphism groups of VOAs. In particular, there is a general result of independent interest, on embedding lattices into unimodular lattices so as to respect automorphism groups and definiteness.

*Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1109 USA. E-mail: `rlgmath.lsa.umich.edu`.

†Mathematisches Institut, Universität Freiburg, Eckerstraße 1, 79104 Germany. E-mail: `gerald@mathematik.uni-freiburg.de`.

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Contents

Notation and terminology

$Aut(V)$	The automorphism group of the VOA V.
$\mathcal{C} = \mathcal{C}(F)$	The binary code determined by the T_r-module structure of V^0.
$\mathcal{D} = \mathcal{D}(F)$	The binary code of the $I \subseteq \{1, \ldots, r\}$ with $V^I \neq 0$.
$\Delta \cong L/M$	A $\mathbb{Z}_4$-code associated to a lattice L with fixed frame sublattice M.
D_X	The normal subgroup of W_X which stabilizes each subset $\{\pm x\}$ of X.
E_8	The root lattice of $E_8(\mathbb{C})$.
$E_8(\mathbb{C})$	The Lie group of type E_8 over the field of complex numbers.
$\eta(h) = exp(2\pi i h_0)$	For $h \in V_1$, $\eta(h)$ is an automorphism of the VOA V.
$F = \{\omega_1, \ldots, \omega_r\}$	A Virasoro frame.
FVOA	Abbreviation for framed vertex operator algebra.
$G = G(F)$	The subgroup of $Aut(V)$ fixing the VF F of V.
$G_{\mathcal{C}} = G_{\mathcal{C}}(F)$	The normal subgroup of $G(F)$ acting trivially on T_r.
$G_{\mathcal{D}} = G_{\mathcal{D}}(F)$	The normal subgroup of $G(F)$ acting trivially on V^0.
H_8, H_{16},	The Hamming codes of length 8 resp. 16.
$\mathfrak{h}$	A Cartan subalgebra in V_L
k	The dimension of $\mathcal{D}$.
$L,\ L^*$	An integral lattice of rank n, often self-dual and even, and its dual.
L_C	The even lattice constructed from a doubly-even code C.
M	A frame sublattice of L. This is a sublattice isomorphic to D_1^n
$M(h_1, \ldots, h_r)$	The irreducible T_r-module of highest weight $(h_1, \ldots, h_r) \in \{0, \frac{1}{2}, \frac{1}{16}\}^r$.
N	The normalizer of T in $Aut(V_L)$.
r	The number of elements in a VF.
t	The Miyamoto map $F \longrightarrow G_{\mathcal{D}}(F)$.
T	A toral subgroup of $Aut(V_L)$ for integral even lattice L with a frame sublattice M.
$T_r = M(0)^{\otimes r}$	The tensor product of r simple Virasoro VOAs of rank $\frac{1}{2}$.
V	An arbitrary VOA, or the VOA V_{E_8}.
V_L	The VOA constructed from an even lattice L.
VF	Abbreviation for Virasoro frame.
VOA	Abbreviation for vertex operator algebra.
$V_{\mathfrak{h}}$	The canonical irreducible module for the Heisenberg algebra based on the finite dimensional vector space $\mathfrak{h}$.
V^I	The sum of irreducible T_r-submodules of V isomorphic to $M(h_1, \ldots, h_r)$ with $h_i = \frac{1}{16}$ if and only if $i \in I$.
$V^0 = V^{\emptyset}$	This is V^I, for $I = \emptyset$.
W_X	The stabilizer of a lattice frame X of a lattice L in $Aut(L)$.
$Y(\,.\,, z)$	A vertex operator.
$x(-n)$	Abbreviation for the element $t^{-n} \otimes x$ in V_L.
X	A lattice frame of L. These are the vectors of norm 4 in a frame sublattice.

1 Introduction

In this article, we determine, up to automorphisms, the Virasoro frames and their stabilizers for V_{E_8}, the lattice type vertex operator algebra based on the E_8-lattice.

In [DGH], the basic theory of framed vertex operator algebras (FVOAs) was established. It included some general structure theory of frame stabilizers, the subgroup of the automorphism group fixing the frame setwise. It is a finite group with a normal 2-subgroup of class at most 2 and quotient group which embeds in the common automorphism group of a pair of binary codes. There was no procedure for computing the exact structure. To develop our understanding of FVOA theory, we decided to settle the frame stabilizers definitively for the familiar example V_{E_8}. This result, with further general theory for analyzing frame stabilizers in lattice type FVOAs, is presented in this article. Even simple questions such as whether the C-group (defined below) can be nonabelian or of exponent greater than 2 did not seem answerable with the techniques in [DGH] (the C-groups for V_{E_8} turn out to be nonabelian for four of the five orbits and elementary abelian for the last orbit). Furthermore, in V_{E_8}, we also show that there are just five orbits on frames, a point which was left unsettled in [DGH].

The study of FVOAs is a special case of the general extension problem of nice rational VOAs. The problem can be formulated completely in terms of the associated modular braided tensor category or 3d-TQFT (cf. [H] and the introduction of [DGH]). There has been recent progress in this direction [B, M99], proving also conjectures from [FSS], but a general theory for such extensions is unknown, even for FVOAs. The analysis of Virasoro frame stabilizers contributes to this problem by computing the automorphisms of such extensions. Furthermore, our classification result for the five VFs in V_{E_8} can be used to show the uniqueness of the unitary self-dual VOA of central charge 24 with Kac-Moody subVOA $V_{A_{1,2}}^{\otimes 16}$ (cf. [DGH], Remark 5.4) since up to roots of unity the associated modular braided tensor category is equivalent to the one for the Virasoro subVOA $L_{1/2}(0)^{\otimes 16}$ (cf. [MS]). This seems to be the first uniqueness result for one of the 71 unitary self-dual VOA candidates of central charge 24 given by Schellekens [Sc] which is not the lattice VOA of a Niemeier lattice.

Before stating our main results, we review some material about Virasoro frames from [DGH].

A subset $F = \{\omega_1, \ldots, \omega_r\}$ of a simple vertex operator algebra (VOA) V is called a *Virasoro frame* (VF) if the ω_i for $i = 1, \ldots, r$ generate mutually commuting simple Virasoro vertex operator algebras of central charge $1/2$ and $\omega_1 + \cdots + \omega_r$ is the Virasoro element of V. Such a VOA V is called a *framed vertex operator algebra* (FVOA).

We use the notation of [DGH] throughout. In particular, we shall use G for the stabilizer of the Virasoro frame F in the group $Aut(V)$. There are two binary codes, $\mathcal{C}$, $\mathcal{D}$ and we use k for the integer $dim(\mathcal{D})$. There will be some obvious

modifications of the [DGH] notation, such as $\mathcal{D}(F)$ to indicate dependence of the code $\mathcal{D}$ on the Virasoro frame F, $G(F)$, $G_{\mathcal{D}}(F)$, $G_{\mathcal{C}}(F)$, etc. We call the group $G_{\mathcal{D}}$ the *D-group of the frame* and we call $G_{\mathcal{C}}$ the *C-group of the frame* (see [DGH], Def. 2.7).

Denote for an abelian group A with $\widehat{A} = Hom(A, \mathbb{C}^{\times})$ the dual group. Throughout this paper, we use standard group theoretic notation [Go, H]. For instance, if J is a group and S a subset, $C(S)$ or $C_J(S)$ denotes the centralizer of S in J, $N(S)$ or $N_J(S)$ denotes the normalizer of S in J and $Z(J)$ denotes the center of J.

We summarize the basic properties of G.

Proposition 1.1 *(i) $G_{\mathcal{D}} \leq G_{\mathcal{C}}$ and $G_{\mathcal{D}}$ and $G_{\mathcal{C}}/G_{\mathcal{D}}$ are elementary abelian 2-groups;*

(ii) $G_{\mathcal{D}} \leq Z(G_{\mathcal{C}})$;

(iii) $G_{\mathcal{D}} \cong \widehat{\mathcal{D}}$ and $G_{\mathcal{C}}/G_{\mathcal{D}}$ embeds in $\widehat{\mathcal{C}}$;

(iv) G is finite, and the action of G on the frame embeds $G/G_{\mathcal{C}}$ in Sym_r.

Proof. [DGH], Th. 2.8. The assertion $G_{\mathcal{D}} \leq Z(G_{\mathcal{C}})$ is easy to check from the definitions, but unfortunately was not made explicit in [DGH]. □

We have that $G_{\mathcal{C}}/G_{\mathcal{D}}$ embeds in $\widehat{\mathcal{C}}$, but general theory has not yet given a definitive description of the image.

We summarize our main results below. See Section 2 for certain definitions. Note that Main Theorem I (ii) just refers to the text for methods.

Main Theorem I

(i) In the case of a lattice type VOA based on a lattice L and a VF which is associated to a lattice frame, X, we have a description of $G \cap N$, where N is the normalizer of a natural torus T (see Th. 2.8). It is an extension of the form $(G \cap T).W_X$, where W is the automorphism group of the lattice and W_X is the stabilizer in W of X. Let D_X be the subgroup of W_X which stabilizes each set $\{\pm x\}$, for $x \in X$. Let $n = rank(L)$ and suppose that L is obtained from the sublattice spanned by X by adjoining "glue vectors" forming the $\mathbb{Z}_4$-code $\Delta \cong 2^{\ell} \times 4^k$. We have $G_{\mathcal{C}} \leq N$, $G_{\mathcal{C}} \cap T \cong 2^{\ell} \times 4^k$, $G_{\mathcal{C}}/(G_{\mathcal{C}} \cap T) \cong D_X$ and $G \cap T \cong 2^{n-\ell-k} \times 4^{\ell} \times 8^k$.

(ii) Assume that in the situation (i) the lattice comes from a marking of a binary code. Then a triality automorphism σ is defined (cf. [DGH], after Theorem 4.10) and one has $G \geq \langle G \cap N, \sigma \rangle > G \cap N$. In particular the group of permutations induced on the VF by $\langle G \cap N, \sigma \rangle$ strictly contains the group induced by $G \cap N$. We give conditions for identifying these permutation groups. In the case of V_{E_8}, the cases $dim(\mathcal{D}) = 1$, 2 and 3 come from a marking and we prove that $\langle G \cap N, \sigma \rangle = G$.

Main Theorem II *Let V be the lattice VOA based on the E_8-lattice.*

(i) There are exactly five orbits for the action of $Aut(V) \cong E_8(\mathbb{C})$ on the set of VFs in V.

(ii) These five orbits are distinguished by the parameter k, the dimension of the code $\mathcal{D}$, and in these respective cases $G = G(F)$, the stabilizer of the Virasoro frame F, has the following structure:

k	G
1	$2^{1+14}Sym_{16}$
2	$2^{2+12}[Sym_8 \wr 2]$
3	$[2^{3+9} \cong 2^{4+8}]2^8[Sym_3 \wr Sym_4] \cong$ $[2^{3+9} \cong 2^{4+8}][Sym_4 \wr Sym_4] \cong$ $2^{4+16}[Sym_3 \wr Sym_4]$
4	$2^{4+5}[2 \wr AGL(3,2)] \cong [2^4 \times 8^4]2.AGL(3,2)$
5	$2^5 AGL(4,2) \cong 4^4[2{\cdot}GL(4,2)]$

(iii) In these five cases, the frame stabilizers G are determined up to conjugacy as subgroups of $E_8(\mathbb{C})$ by group theoretic conditions. Sets of conditions which determine them are found in Section 4 and listed below for each k.

$k = 1$: *G is the normalizer of the unique up to conjugacy subgroup isomorphic to 2_+^{1+14}; equivalently, the unique up to conjugacy subgroup isomorphic to $2_+^{1+14}Sym_{16}$.*

$k = 2$: *G satisfies the hypotheses of this conjugacy result: In $E_8(\mathbb{C})$, there is one conjugacy class of subgroups which are a semidirect product $X\langle t\rangle$, where t has order 2, $X = X_1X_2$ is a central product of groups of the form $[2 \times 2^{1+6}]Sym_8$ such that $X_1 \cap X_2 = Z(X_1) = Z(X_2)$ and conjugation by t interchanges X_1 and X_2.*

$k = 3$: *G is a subgroup of $E_8(\mathbb{C})$ characterized up to conjugacy as a subgroup X satisfying the following conditions:*

(a) X has the form $[2^{3+9} = 2^{4+8}]Sym_4^4.Sym_4 \cong [2^{4+16}]Sym_3 \wr Sym_4$;

(b) X has a normal subgroup $E \cong 2^3$ which is $2B$-pure.

$k = 4$: *G has the form $[2^4 \times 8^4]2.AGL(3,2) \cong 2^{4+5+8}AGL(3,2)$ and is characterized up to conjugacy by this property: it is contained in a subgroup G_1 of the form $[2^4 \times 8^4][2{\cdot}GL(4,2)]$, which is uniquely determined in $E_8(\mathbb{C})$ up to conjugacy in the normalizer of a $GL(4,2)$-signalizer (defined in Section 4.4); in particular, G is determined uniquely up to conjugacy in G_1 as the stabilizer of a subgroup isomorphic to 2^3 in the $GL(4,2)$-signalizer.*

$k = 5$: *G is conjugate to a subgroup of the Alekseevski group (see [A, G76] and Prop. 3.5) of the form $2^5AGL(4,2) \cong 4^4[2{\cdot}GL(4,2)]$ and the set of all such subgroups of the Alekseevski group form a conjugacy class in the Alekseevski group.*

Remark 1.2 We stress that there are two main viewpoints to the analysis in this article. One is group structures coming from binary codes and lattices via markings and frames as in [DGH]; and the other is the theory of finite subgroups of $E_8(\mathbb{C})$ (for a recent survey, see [GR99]).

Appendix 5.1 contains a proof that, given an even lattice L of signature (p, q), there is an integer $m \leq 8$ so that L embeds as a direct summand of a unimodular even lattice M of signature (mp, mq) and so that $Aut(M)$ contains a subgroup which stabilizes L and acts faithfully on L as $Aut(L)$. This is similar in spirit to results of James and Nikulin [J, N] (which display such embeddings into indefinite lattices) and gives a useful containment of VOAs $V_L \leq V_M$.

Appendix 5.2 is a construction of a group extension $\widetilde{W}$ of the automorphism group W of a lattice L by an elementary abelian 2-group. This extension plays a natural role in the automorphism group of V_L. While this construction is not new, it it useful to make things explicit for certain proofs in this article. Also, there are some historical remarks.

Appendix 5.3 discusses the group extension aspect of the frame stabilizers. At first, it looks like the groups $G/G_{\mathcal{D}}$ might split over $G_{\mathcal{C}}/G_{\mathcal{D}}$, but some do not.

Appendix 5.4 is a technical result about permutation representations for a classical group.

2 Stabilizers for framed lattice VOAs

In [DGH], we used the following concept.

Definition 2.1 A *lattice frame* in a rank n lattice $L \leq \mathbb{R}^n$ is a set, X, of $2n$ lattice vectors of squared length 4 in L such that two elements are equal, opposite or orthogonal. Every lattice frame spans a lattice $M \cong D_1^n$, called the *frame sublattice*.

Clearly, in a given lattice, there is a bijection between lattice frames and frame sublattices (the frame defining the frame sublattice is the set of minimal vectors in that sublattice). Note that in [DGH] the term lattice frame means sublattice.

In Chapter 3 of [DGH], we constructed, for every integral lattice containing a frame sublattice, a Virasoro frame for the associated rank n lattice VOA and determined the decomposition into modules for the Virasoro subVOA T_{2n} belonging to this Virasoro frame. In Section 2.1, we will determine the subgroup of the Virasoro frame stabilizer which is visible from this construction. As in [DGH], we will use the language of $\mathbb{Z}_4$-codes. We also prove a result about the centralizer of $G_{\mathcal{C}}$ for some framed lattices. In Section 2.2, we look at lattices

with frame sublattices constructed from marked binary codes as in Chapter 4 of [DGH]. Here, a triality automorphism is defined; see Theorem 2.19.

2.1 General integral even lattices

Let V_L be the lattice VOA, based on the integral even lattice L. For every frame sublattice M of L there is *the associated VF* $F = \{\omega_1, \ldots, \omega_{2n}\}$ inside $V_M \subset V_L$ (cf. [DGH], Def. 3.2). If X is the lattice frame contained in M, the associated VF F is the set of all $\frac{1}{16}x(-1)^2 \pm \frac{1}{4}(e^x + e^{-x})$, $x \in X$. We use $x(-n)$ for the element $t^{-n} \otimes x$ in V_L.

Using the notation of [DGH], we can describe some structure of the Cartan subalgebra $\mathfrak{h} = t^{-1} \otimes_{\mathbb{C}} (L \otimes_{\mathbb{Z}} \mathbb{C}) \subset V_L$ associated to a frame sublattice of L:

Proposition 2.2 (Cartan subalgebra) *Let M be a frame sublattice spanned by a lattice frame inside an integral even lattice L of rank n and let T_{2n} be the subVOA of $V_M \leq V_L$ generated by the associated Virasoro frame of the lattice VOA V_L. Then*

(i) $\mathfrak{h} = (V_M)_1$ *is an abelian Lie algebra of rank n.*

(ii) It is the n-dimensional highest weight space for the T_{2n}-submodule of V_L isomorphic to the direct sum

$$M(\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0) \oplus M(0, 0, \frac{1}{2}, \frac{1}{2}, 0, \ldots, 0) \oplus \cdots \oplus M(0, \ldots, 0, \frac{1}{2}, \frac{1}{2}).$$

The summands are spanned by vectors of the form $x(-1)$, where x is in the lattice frame.

Proof. For the first statement, recall that as a graded vector space $V_M = V_{\mathfrak{h}} \otimes \mathbb{C}[M]$. Since the minimal nonzero squared length of a vector x in the lattice $M \cong D_1^n$ is 4, i.e. $e^x \in \mathbb{C}[M]$ has conformal weight 2, the weight one part of V_M is just the the weight one part of the Heisenberg VOA $V_{\mathfrak{h}}$, i.e., in the usual notation, $\mathfrak{h} = t^{-1} \otimes_{\mathbb{C}} (M \otimes_{\mathbb{Z}} \mathbb{C})$. It inherits a toral Lie algebra structure from the Lie algebra V_1.

For the second statement use Corollary 3.3. (1) of [DGH]. Since $M(h_1, \ldots, h_{2n})$ has minimal conformal weight $h_1 + \cdots + h_{2n}$ (this is the smallest i so that $M(h_1, \ldots, h_{2n})$ has an $L(0)$-eigenvector for the eigenvalue i) and the weight one part of $M(0, \ldots, 0)$ is zero, the assertion follows. □

Throughout this article, when we work with a VOA based on a lattice with lattice frame, we write $\mathfrak{h}$ for the above Cartan subalgebra $(V_M)_1$ of $(V_L)_1$.

Corollary 2.3 *In the situation where the VF comes from a lattice frame, $G_{\mathcal{C}}$ normalizes the Cartan subalgebra $\mathfrak{h}$ of $(V_L)_1$.*

Proof. We use the proof of Prop. 2.2 and its notation. For every $(h_1, \ldots, h_{2n})$, the group $G_{\mathcal{C}}$ leaves the submodule associated to $M(h_1, \ldots, h_{2n})$ in the Virasoro module decomposition invariant, so it normalizes the Lie algebra $\mathfrak{h} \le (V_L)_1$. $\square$

Definition 2.4 Elements of $(V_L)_1$ act as locally finite derivations under the 0^{th} binary composition on V_L. Such endomorphisms may be exponentiated to elements of $Aut(V_L)$. For $h \in \mathfrak{h}$, we define $\eta(h) := exp(2\pi i h_0)$, so that η is a homomorphism from $\mathfrak{h}$ to $Aut(V_L)$. Let $T := \eta(\mathfrak{h})$ be the associated torus of automorphisms. The scale factor $2\pi i$ gives us the exact sequence

$$0 \longrightarrow L^* \longrightarrow \mathfrak{h} \xrightarrow{\eta} T \longrightarrow 1.$$

Let $N := N(T)$ be the normalizer of the torus T in $Aut(V_L)$ and denote by $\widetilde{W}$ the lift of $W := Aut(L)$ to a subgroup of $Aut(V_L)$, as described in Appendix 5.2. Finally, we need the subgroup $K := \langle exp(2\pi i x_0) \mid x \in (V_L)_1 \rangle$.

Proposition 2.5 *For any lattice VOA we have*

(i) $N = T\widetilde{W}$ *and* $N/T \cong W$*;*

(ii) $Aut(V_L) = KN$ *and* K *is a normal subgroup.*

Proof. Part (i) follows from [DN] and the construction of $\widetilde{W}$ given in Appendix 5.2; part (ii) is due to [DN]. $\square$

Definition 2.6 For a subset X of L, let W_X be its setwise stabilizer. We can identify X as a subset X of V_L via the embedding $L \subset \mathfrak{h}$. Let $\widetilde{W}_X$ indicate the setwise stabilizer of X in $\widetilde{W}$.

When X is a lattice frame, D_X denotes the subgroup of W_X which stabilizes each subset $\{\pm x\}$ of X (so D_X acts "diagonally" with respect to the double basis X). Always, $-1 \in D_X$. Let $\widetilde{D}_X$ be the preimage in $\widetilde{W}$.

Given a lattice frame $X \subset L$ we will describe the intersection $G \cap N$ of the frame group G for the associated VF F with N. By using Prop. 2.2 we will show how to get $G_{\mathcal{C}}$ in the course of studying $G \cap T$ and $G \cap N$.

Definition 2.7 (The code Δ and integers k, ℓ) Recall $n = rank(L)$. Let X be the lattice frame and M the associated sublattice. We observe that $M \le L \le L^* \le M^* = \frac{1}{4}M$ and M determines a $\mathbb{Z}_4$-code $\Delta \le \mathbb{Z}_4^n$ which corresponds to $L/M \le M^*/M$ by the identification $M^*/M \cong \mathbb{Z}_4^n$ extending some $\{\pm 1\}$-equivariant bijection of $\frac{1}{4}X$ with the set of vectors $(0, \ldots, 0, \pm 1, 0, \ldots, 0)$ (cf. [DGH], p. 462).

There are integers ℓ and k such that the code Δ is, as an abelian group, isomorphic to $2^\ell \times 4^k$.

By Th. 4.7 of [DGH], one has $k = dim(\mathcal{D})$. We have $\ell + 2k \leq n$ since L is integral and $\ell + 2k = n$ if and only if L is self-dual. Note that since L contains a frame sublattice, its determinant must be an even power of 2. In terms of the $\mathbb{Z}_4$-code Δ we get for its automorphism group

$$Aut(\Delta) \cong W_X \leq Mon(n, \mathbb{Z}_4) \cong 2^n{:}Sym_n$$

and D_X is a normal subgroup of sign changes in W_X.

Theorem 2.8 (The intersection $G \cap N$) *For the frame stabilizer G and the normal subgroups $G_{\mathcal{C}}$ and $G_{\mathcal{D}}$ we have:*

(i) $G_{\mathcal{D}} \leq T$, $G_{\mathcal{D}} = \eta(\frac{1}{2}M + L^*) \cong (\frac{1}{2}M + L^*)/L^* \cong 2^k$;

(ii) $G_{\mathcal{C}} \leq N$, $G_{\mathcal{C}} \cap T \cong \Delta \cong 2^\ell \times 4^k$, $G_{\mathcal{C}}/(G_{\mathcal{C}} \cap T) \cong D_X$;

(iii) $G \cap T \cong 2^{n-\ell-k} \times 4^\ell \times 8^k$, $G \cap N = (G \cap T)\widetilde{W}_X$, $(G \cap N)/G_{\mathcal{C}} \cong 2 \wr (W_X/D_X)$, *where the wreath product is taken with respect to the action of W_X/D_X on the n-set of pairs $\{\pm x\}$, $x \in X$.*

Proof. First, we describe $G \cap T$. The group T acts on the VF, consisting of elements of the form $\frac{1}{16}x(-1)^2 \pm \frac{1}{4}(e^x + e^{-x})$, $x \in X$. A transformation $\eta(h) \in \eta(\mathfrak{h}) = T$ will fix $x(-1)^2$ and send $e^x + e^{-x}$ to $a\, e^x + a^{-1} e^{-x}$, with $a = e^{2\pi i (h,x)}$.

Using [DGH], Th. 4.7, we see that $\eta(h) \in T$ is in $G_{\mathcal{D}}$ if $(h, L \cap \frac{1}{2}M) \leq \mathbb{Z}$, i.e., if $h \in 2M^* + L^* = \frac{1}{2}M + L^*$. Since $\eta(\frac{1}{2}M + L^*) \cong (\frac{1}{2}M + L^*)/L^* \cong 2^k$ has the same order as $\widehat{\mathcal{D}} \cong G_{\mathcal{D}}$, part (i) is proven.

For an element $\eta(h)$ of T to centralize the frame, the requirement is all of the a above equal 1, which is equivalent to $(h, M) \leq \mathbb{Z}$. This defines the set $M^* = \frac{1}{4}M$, and its image is $\eta(M^*) \cong (M^* + L^*)/L^* = M^*/L^* \cong L/M \cong \Delta \cong 2^\ell \times 4^k$.

For $\eta(h) \in T$ to stabilize the frame, the requirement is that all $a \in \{\pm 1\}$, which is equivalent to $(h, M) \leq \frac{1}{2}\mathbb{Z}$, i.e., $h \in \frac{1}{2}M^* = \frac{1}{8}M$. The image of $\frac{1}{8}M$ under η is isomorphic to $(\frac{1}{8}M + L^*)/L^* \cong 2^{n-\ell-k} \times 4^\ell \times 8^k$. The first part of (iii) follows.

For the rest of (ii), one has $G_{\mathcal{C}} \leq N$ by Corollary 2.3. Recall notations from Def. 2.6. It is clear that $G_{\mathcal{C}} \leq (G \cap T)\widetilde{D}_X$. The action of $G \cap T$ on the VF implies that $G_{\mathcal{C}}$ meets every coset of $G \cap T$ in $(G \cap T)\widetilde{D}_X$, i.e., $(G \cap T)\widetilde{D}_X = (G \cap T)G_{\mathcal{C}}$, whence $G_{\mathcal{C}}/(G_{\mathcal{C}} \cap T) \cong D_X$.

The last two statements of (iii) follow from (ii), the structure of $G \cap T$ and Proposition 5.10 (ii). In more detail, $G \cap N$ acts on X and on F. In fact, there is the $G \cap N$-equivariant map $F \longrightarrow X/\{\pm 1\}$ by $\frac{1}{16}x(-1)^2 \pm \frac{1}{4}(e^x + e^{-x}) \mapsto \{\pm x\}$, for $x \in X$. Now, on $X/\{\pm 1\}$, $G \cap N$ induces W_X/D_X. The kernel modulo $G_{\mathcal{C}}$ is at most a group of order 2^n, where $n = |X/\{\pm 1\}|$. On the other hand,

$G \cap T$ induces a group of this order on X and it is in the kernel of this action. Consider the partition P of X into the n pairs $\{x, -x\}$, $x \in X$. The stabilizer of P in Sym_X is a wreath product $2 \wr Sym_n$, where the normal 2^n subgroup is the kernel of the action on the n-set P. By the Dedekind lemma, any subgroup H of $Stab(P)$ which contains this normal subgroup is a semidirect product, and this remark applies to the action of our group $G \cap N$ on X. □

From the above proof, we get these observations.

Corollary 2.9 *The exponent of the group $G \cap T$ is 8 if $k \geq 1$. The order of $G_{\mathcal{C}}$ is $2^{\ell+2k+e}$, where $2^e = |D_X|$. Since $-1 \in D_X$, $e \geq 1$. See Tables 2 and 3 in Section 3 for certain values of k and e when $V_L = V_{E_8}$.*

In fact, one can consider $G_{\mathcal{C}}$ as a group extension in several ways. The above discussion shows that $G_{\mathcal{C}}$ is an extension of the abelian group $G_{\mathcal{C}} \cap T$ and a group isomorphic to D_X. On the other hand, $G_{\mathcal{C}}$ has the central elementary abelian subgroup $G_{\mathcal{D}} \cong 2^k$ giving elementary abelian quotient $2^{\ell+k+e}$ (cf. Prop. 1.1).

We next describe the centralizer of $G_{\mathcal{C}}$; to this end, we need some background from the theory of Chevalley groups.

Recall that, by definition, a Chevalley group over a field is contained in the automorphism group of a Lie algebra and has trivial center. The associated Steinberg group is a central extension of the Chevalley group and has the property that the preimage of a Cartan subgroup is abelian. See [Ca].

In the next result, we formally extend the "normal form" (or "canonical form") concept for Steinberg groups to a wider class of groups which includes automorphism groups of lattice type VOAs. The notation of [Ca] is fairly standard: G, H, N, B, U stand for a Steinberg group, a Cartan subgroup, a subgroup of the normalizer of the Cartan subgroup, a Borel subgroup containing H, the normal subgroup of B generated by root groups. These groups participate in the so-called (B, N) pair and normal form structures for G.

In this article, the symbols G and N have already been assigned meanings, so we shall deviate from traditional Chevalley theory notations slightly for the next two results. We shall use notations R, H_R, N_R, B_R, U_R for a group R with subgroups H_R, N_R, B_R, U_R of R which will have properties like G, H, N, B, U. In an algebraic group R, H_R stands for a maximal torus. If R is a torus, $H_R = N_R = R$ and $U_R = 1$. If R is an algebraic group and the connected component of the identity R^0 is a torus, then $H_R = R^0$, $N_R = R$, $U_R = 1$. If R is a reductive algebraic group, H_R is a maximal torus and N_R is its normalizer. In Steinberg groups, the group N_R is not necessarily the full normalizer of H_R and $H_R = 1$ is possible (for small fields).

The automorphism groups of lattice type VOAs are reductive algebraic groups over the complex numbers, and so are an instance of the group R of Lemma 2.10 (iii) with R^0 the connected component of the identity and N_R the

normalizer of a maximal torus, usually denoted by the symbol T instead of H (these subgroups lie in R^0).

Lemma 2.10 *(i) In a Steinberg group R over a field, every element may be written uniquely in the "normal form" uhn_wu', where, in usual notation,*

- *$h \in H_R$, a Cartan subgroup;*
- *N_R is a subgroup containing H_R as a normal subgroup and for each w in N_R/H_R, there is a choice of preimage n_w in N_R;*
- *$u \in U_R$, the subgroup generated by root groups for the set of positive roots;*
- *$u' \in U_{R,w}$, the subgroup of U_R generated by root groups associated to positive roots r such that $w(r)$ is negative.*

(ii) We have a normal form as in (i) for direct products of Steinberg groups and tori, and even for central products of universal Steinberg groups and tori.

Suppose that $R = R_1 \cdots R_n$ is a central product where R_i is a torus or a Steinberg group, all over a common field. To each index i is associated a sequence R_i, H_{R_i}, N_{R_i}, B_{R_i}, U_{R_i} for which we have a normal form as in (i). Then we have a normal form for R given by the sequence R, H_R, N_R, B_R, U_R, where $R = R_1 \cdots R_n$ and H_R is the product of the H_{R_i} and similarly for N_R, B_R, U_R.

(iii) We have a normal form as in (i) for groups R which contain a normal subgroup R_0 which is a direct product as in (ii) such that there exists a subgroup N_R of R satisfying $N_R \cap R_0 = N_{R_0}$. It suffices to take $H_R = H_{R_0}$, $U_R = U_{R_0}$, $B_R = B_{R_0}$.

Proof. For (i), see [Ca]. Parts (ii) and (iii) are formal. $\square$

Corollary 2.11 *Suppose that R is a group as in Lemma 2.10 (iii). Let S be a subset of the Cartan subgroup H_R. Then $C_R(S) = EC_{N_R}(S)$, where E is generated by all the root groups centralized by S.*

Proof. Let $g \in R$ and $s \in S$. Consider $g = uhn_wu'$ in normal form and study the conjugate ${}^sg := sgs^{-1} = {}^su{}^sh{}^sn_w{}^su' = {}^suh{}^sn_w{}^su'$. Observe that the Cartan subgroup normalizes each root group, hence also U_R and $U_{R,w}$, as in Lemma 2.10. If we write ${}^sn_w = h_1n_w$, for an element $h_1 \in H$, then we get the normal form ${}^su(hh_1)n_w \cdot {}^su'$ for sgs^{-1}. Therefore, g commutes with s if and only if u, u' and n_w commute with s. $\square$

Note that the intersection hypothesis of Lemma 2.10 (iii) and $R = R^0N_R$ is satisfied by complex reductive algebraic groups, in particular by $Aut(V_L)$. (R^0 is the connected component of the identity and N_R is the normalizer in R of a maximal torus.)

Corollary 2.12 *Let S be a subset of T. Then $C_{Aut(V_L)}(S) = EC_N(S)$, where E is generated by all the root groups in K centralized by S.*

Definition 2.13 In a torus T, let $T_{(m)} = \{u \in T \mid u^m = 1\} \cong \mathbb{Z}_m^{rank(T)}$, for an integer $m > 0$.

Corollary 2.14 *(i) $C_T(G_{\mathcal{C}}) \leq T_{(2)}$.*

(ii) $G'_{\mathcal{C}} \neq 1$.

(iii) $C_{Aut(V_L)}(G_{\mathcal{C}}) \leq C_{Aut(V_L)}(G_{\mathcal{C}} \cap T) \leq N$.

(iv) $TC_{Aut(V_L)}(G_{\mathcal{C}})/T$ corresponds to a subgroup of W_X under the identification of N/T with W.

Proof. Since $G_{\mathcal{C}}$ contains an element u corresponding to $-1 \in D_X$, we have $C_T(G_{\mathcal{C}}) \leq C_T(u) = T_{(2)}$, proving (i).

For the proof of (ii), note that $G_{\mathcal{C}} \cap T$ has exponent 4.

For (iii), recall that $G_{\mathcal{C}} \cap T = \eta(M^*)$. Since $\eta(M^*) \leq T$, a maximal torus, we use Lemma 2.12 to deduce that the centralizer of $\eta(M^*)$ in $Aut(V_L)$ is $EC_N(\eta(M^*))$, where E is generated by all the root groups with respect to T which are centralized by $\eta(M^*)$. There are no such root groups since $(M^*)^* = M$, which contains no roots. So, $E = 1$.

For (iv), just observe that $C_{Aut(V_L)}(G_{\mathcal{C}})$ is a subgroup of N, and leaves invariant $\eta(M^*)$. So, in its action on $\mathfrak{h}$, it preserves M^* and M, hence also X. □

Theorem 2.15 *Suppose that L^* contains no nonzero elements of $\frac{1}{4}X + \frac{1}{4}X$ (e.g., this holds if L^* has no vectors of squared length $\frac{1}{2}$ and 1). Then $C_{Aut(V_L)}(G_{\mathcal{C}}) \leq T_{(2)}$.*

Proof. Let $C := C_{Aut(V_L)}(G_{\mathcal{C}})$. Use Cor. 2.14 (iv) and suppose that $c \in C$ corresponds to a nonidentity element of W_X. Then there is $x \in X$ so that $cx \neq x$. Since c is trivial on $G_{\mathcal{C}} \cap T = \eta(M^*) \cong M^*/L^*$, we get $\frac{1}{4}x - \frac{1}{4}cx \in L^* \setminus \{0\}$, a contradiction. So, $C \leq T$. From Cor. 2.14 (i), we get $C = C_T(G_{\mathcal{C}}) \leq T_{(2)}$. □

Corollary 2.16 *$C_{Aut(V_L)}(G_{\mathcal{C}}) \leq T_{(2)}$ if L is self-dual.*

Proof. Since $L = L^*$, every element of L^* has even integer norm, so this is obvious. □

Remark 2.17 (i) When the conclusion of Theorem 2.15 holds, the subgroup $C_{Aut(V_L)}(G_{\mathcal{C}})$ of $T_{(2)}$ depends just on the action of D_X on $T_{(2)} \cong \frac{1}{2}L^*/L^*$.

(ii) In the E_8 lattice example, $C_{E_8(\mathbb{C})}(G_{\mathcal{C}}) \leq T_{(2)}$, by Cor. 2.16. For $k = 1$, $C_{E_8(\mathbb{C})}(G_{\mathcal{C}})$ is contained in $G_{\mathcal{C}}$; for $k = 4$, it is not contained in $G_{\mathcal{C}} \cong 4^4{:}2$ (in this case, $C_{E_8(\mathbb{C})}(G_{\mathcal{C}}) = T_{(2)}$). See Section 4.

2.2 Lattices from marked binary codes

In the more special situation where a VOA is constructed with the help of binary codes there exists a so-called triality automorphism σ. The triality automorphism was first defined in [G82] as an automorphism of the Griess algebra and in [FLM] it was extended to an automorphism of the Moonshine module $V^\natural$. In [DGM, DGM2], it was shown that for any doubly even self-dual code C one can define σ for both of the VOAs V_{L_C} and $\widetilde{V}_{\widetilde{L}_C}$ (cf. [DGH], Sect. 4, for the notation).

For binary codes we introduced in [DGH] the notation of a marking.

Definition 2.18 Let n be even. A *marking* of a length n binary code C is a partition of the n coordinates into $n/2$ sets of size 2.

The binary code is a subspace of $\mathbb{F}_2^n$, but it may be considered as a subset of $\mathbb{C}^n$ by interpreting the coordinates 0 and 1 modulo 2 as the ordinary integers 0 and 1.

For a binary code there is the lattice

$$L_C = \{\frac{1}{\sqrt{2}}(c+x) \mid c \in C,\ x \in (2\mathbb{Z})^d\}.$$

The lattice L_C is integral and even if the code C is doubly-even. Every marking of the binary code C determines a frame sublattice inside L_C (cf. [DGH], p. 425).

Again, let F be the VF associated to this lattice frame.

Theorem 2.19 (cf. [DGH], p. 432) *For the VF F inside the lattice VOA V_{L_C} constructed from a marked binary doubly-even self-dual code C, there is a triality automorphism σ inside $G(F)$.*

Its image $\bar\sigma \in G/G_{\mathcal{C}} \leq Sym_F \cong Sym_{2n}$ interchanges the Virasoro elements ω_{4i-2} and ω_{4i-1} for $i = 1, \ldots, n/2$ and fixes the others.

Proof. The existence of σ was proven [FLM], Th. 11.2.1, and [DGM, DGM2].

The description is the following (see [FLM], (11.1.72), and [DGM]): Let $A_1 = \sqrt{2}\mathbb{Z}$ be the root lattice of $SL_2(\mathbb{C})$. The group $SL_2(\mathbb{C})$ acts on the lattice VOA V_{A_1} and its module $V_{A_1+1/\sqrt{2}}$. On

$$(V_{A_1})_1 \cong \mathbb{C}\, x(-1) \oplus \mathbb{C}\, e^x \oplus \mathbb{C}\, e^{-x} \cong \mathfrak{sl}_2(\mathbb{C}),$$

where x is a generator of the A_1 root lattice, let $\bar\mu \in PSL_2(\mathbb{C}) \cong SO_3(\mathbb{C})$ be the linear map defined by

$$\bar\mu(x(-1)) = e^x + e^{-x}, \quad \bar\mu(e^x + e^{-x}) = x(-1) \quad \text{and} \quad \bar\mu(e^x - e^{-x}) = -(e^x - e^{-x}).$$

Let μ be one of the two elements in $SL_2(\mathbb{C})$ which maps modulo the center to $\bar{\mu}$. On $V_{L_C} \cong \bigoplus_{c\in C} V_{A_1+c_1/\sqrt{2}} \otimes \cdots \otimes V_{A_1+c_n/\sqrt{2}}$, there is the tensor product action of the direct product of n copies of $SL_2(\mathbb{C})$. The triality automorphism is defined as the diagonal element $\sigma = (\mu, \ldots, \mu) \in SL_2(\mathbb{C}) \times \cdots \times SL_2(\mathbb{C})$ in it. For $n/8$ odd, the definition has to be adjusted by replacing the last component by $\kappa\mu$, where κ is the nontrivial central element of $SL_2(\mathbb{C})$.

Since the code C is even, the action of σ on V_{L_C} is independent of the choice for μ (cf. [FLM], Remark 11.2.3). One has $\sigma^2 = 1$.

Now we describe how σ acts on the Virasoro frame $F \subset V_{A_1^n}$ for the standard marking $\{(1,2), (3,4), \ldots, (n-1,n)\}$: For $i = 1, \ldots, n/2$, the Virasoro elements are

$$\begin{aligned}
\omega_{4i-3} &= \tfrac{1}{16}(x_{2i-1}(-1) + x_{2i}(-1))^2 + \tfrac{1}{4}(e^{x_{2i-1}+x_{2i}} + e^{-x_{2i-1}-x_{2i}}),\\
\omega_{4i-2} &= \tfrac{1}{16}(x_{2i-1}(-1) + x_{2i}(-1))^2 - \tfrac{1}{4}(e^{x_{2i-1}+x_{2i}} + e^{-x_{2i-1}-x_{2i}}),\\
\omega_{4i-1} &= \tfrac{1}{16}(x_{2i-1}(-1) - x_{2i}(-1))^2 + \tfrac{1}{4}(e^{x_{2i-1}-x_{2i}} + e^{-x_{2i-1}+x_{2i}}),\\
\omega_{4i} &= \tfrac{1}{16}(x_{2i-1}(-1) - x_{2i}(-1))^2 - \tfrac{1}{4}(e^{x_{2i-1}-x_{2i}} + e^{-x_{2i-1}+x_{2i}}),
\end{aligned}$$

where x_i is the generator of the i-th component of the lattice A_1^n. The action of $SL_2(\mathbb{C})_{2i-1} \times SL_2(\mathbb{C})_{2i}$ on $x_{2i-1}(-1)^2$ and $x_{2i}(-1)^2$ is trivial and on the vector space spanned by the nine elements

$$x_{2i-1}(-1)x_{2i}(-1), \quad x_{2i-1}(-1)e^{x_{2i}}, \quad \ldots, \quad e^{-x_{2i-1}}e^{-x_{2i}} = e^{-x_{2i-1}-x_{2i}}$$

it is the tensor product action of the adjoint action of both factors. The remaining factors $SL_2(\mathbb{C})_j$ act trivially. It follows immediately by computation that σ fixes ω_{4i-3} and ω_{4i} and interchanges ω_{4i-2} with ω_{4i-1}. $\square$

Remark 2.20 A Virasoro frame coming from a lattice frame alone may not have a triality.

3 General results about Virasoro frames in V_{E_8}

From now on, let V be the E_8 lattice VOA, so one has $Aut(V) = E_8(\mathbb{C})$ (cf. [DN]). In [DGH], we found the five Virasoro frames Γ, Σ, Ψ, Θ and Ω inside V. They were constructed with the help of frames in the root lattice E_8 and markings of the binary Hamming code H_8 of length 8 as follows:

There are, up to automorphisms, three markings of the Hamming code H_8, denoted by α, β, γ (Th. 5.1 of [DGH]). They give the three frame sublattices $\mathcal{K}_8$, $\mathcal{K}_8'$ and $\mathcal{L}_8$ inside the E_8 lattice. The final frame $\mathcal{O}_8$ can be obtained by a twisted construction from γ (see Th. 5.2. of [DGH]). The four VFs Γ, Σ, Ψ, and Θ come from the frames $\mathcal{K}_8$, $\mathcal{K}_8'$, $\mathcal{L}_8$ and $\mathcal{O}_8$. The fifth VF Ω is obtained by a twisted construction from $\mathcal{O}_8$ (see Th. 5.3. of [DGH]). Table 1 summarizes this. In it, the arrow $\swarrow$ (resp. $\searrow$) denotes the untwisted (resp. twisted) construction.

Table 1: The markings of H_8, resp. frames of E_8 and V and their relations

object	marking/frame								
H_8			α		β		γ		
			$\swarrow\ \searrow$		$\swarrow\ \searrow$		$\swarrow\ \searrow$		
E_8		$\mathcal{K}_8$		$\mathcal{K}_8'$		$\mathcal{L}_8$		$\mathcal{O}_8$	
		$\swarrow\ \searrow$		$\swarrow\ \searrow$		$\swarrow\ \searrow$		$\swarrow\ \searrow$	
V	Γ		Σ		Ψ		Θ		Ω

Table 2: The frame sublattices in the E_8 lattice

orbit	origin	Type of Δ	D_X	W_X
$\mathcal{K}_8$	α	$2^6 \times 4^1$	2^7	$2^7{:}Sym_8$
$\mathcal{K}_8'$	β, $\widetilde{\alpha}$	$2^4 \times 4^2$	2^6	$2^6.(Sym_4 \wr 2)$
$\mathcal{L}_8$	γ, $\widetilde{\beta}$	$2^2 \times 4^3$	2^4	$2^4.(2 \wr Sym_4)$
$\mathcal{O}_8$	$\widetilde{\gamma}$	4^4	2	$2.AGL(3,2)$

In [DGH], Th. 5.3, we showed further that the possible values of $k = dim(\mathcal{D})$ for any VF are 1, 2, 3, 4, 5 and that they occur for the VFs Γ, Σ, Ψ, Θ, and Ω, in this order. For the values $k \in \{1, 2, 3, 4\}$, the VF is unique up to automorphisms of V. For $k = 5$ this was a conjecture, now proven in Theorem 4.15.

To apply the general discussion about $G \cap N$ from the last section, we summarize in Table 2 again the necessary information about the four frame sublattices and the associated $\mathbb{Z}_4$-code Δ already given in [DGH], Th. 5.2, and [CS].

This table together with Theorem 2.8 proves the next theorem.

Theorem 3.1 *For $k = 1, 2, 3,$ and 4 the structure of $G_{\mathcal{C}}$ and $(G \cap N)/G_{\mathcal{C}}$ is given in Table 3.*

The groups $G_{\mathcal{D}}$ are well known subgroups of $Aut(V) \cong E_8(\mathbb{C})$, see Prop. 3.5.

Table 3: Structural Information about $G \cap N$

k	Structure of $G_{\mathcal{C}}$	$(G \cap N)/G_{\mathcal{C}}$	$d = 2^{5-k}$
1	$2^{1+14} \cong (2^6 \times 4).2^7$	$2 \wr Sym_8$	16
2	$2^{2+12} \cong (2^4 \times 4^2).2^6$	$2 \wr [Sym_4 \wr 2]$	8
3	$2^{3+9} \cong (2^2 \times 4^4).2^4$	$2 \wr [2 \wr Sym_4]$	4
4	$2^{4+5} \cong 4^4.2$	$2 \wr AGL(3,2)$	2

Table 4: Centralizers and Normalizers of $2B$-pure elementary abelian subgroups of $E_8(\mathbb{C})$

k	Centralizer	Normalizer
1	$HSpin(16,\mathbb{C})$	$HSpin(16,\mathbb{C})$
2	$2^2D_4^2{:}2$	$2^2D_4^2{:}[2\times Sym_3]$
3	$2^4A_1^8$	$2^4A_1^8.AGL(3,2)$
4	$T_8.2^{1+6}$	$T_8.2^{1+6}.GL(4,2)$
5	2^{5+10}	$2^{5+10}GL(5,2)$

For general background on subgroups of $E_8(\mathbb{C})$, see [CG] and the recent survey [GR99].

Proposition 3.2 (Involutions in $E_8(\mathbb{C})$) *In the group $E_8(\mathbb{C})$, there are two classes of involutions, denoted $2A$ and $2B$. Their respective centralizers are connected groups of types A_1E_7, D_8, and in more detail, their isomorphism types are $HSpin(16,\mathbb{C})$, $2A_1E_7$, which is a nontrivial central product of a fundamental $SL(2,\mathbb{C})$ subgroup with a simply connected group of type E_7 (the factors have common center of order 2). On the adjoint module, of dimension 248, the spectra are 1^{136}, -1^{112} and 1^{120}, -1^{128}.*

Remark 3.3 Conjugacy classes and centralizers for elements of small orders are discussed in several articles, e.g. [CG, G91].

We recall some information about 2-local subgroups of $E_8(\mathbb{C})$.

Definition 3.4 A subgroup S of a group K is *Y-pure* if all nonidentity elements of S lie in the conjugacy class Y of K. (This definition is often used for elementary abelian p-subgroups of a larger group.)

Proposition 3.5 *For each integer $k = 1, 2, 3, 4$, and 5, there is up to conjugacy a unique $2B$-pure elementary abelian subgroup of rank k in $E_8(\mathbb{C})$. These groups are toral for $k \leq 4$ and nontoral for $k = 5$. Their centralizers and normalizers are described in Table 4.*

Proof. [CG], (3.8); [G91]. □

Let F be any VF in V and let $\mathcal{D}$ the associated binary code. We will identify $G_{\mathcal{D}}$ as a $2B$-pure subgroup in Prop. 3.7.

Proposition 3.6 *The binary code $\mathcal{D}$ is equivalent to a code generated by the first $k = 1, 2, 3, 4, 5$ codewords of the list 1^{16}, 1^80^8, $(1^40^4)^2$, $(1100)^4$, $(10)^8$.*

Proof. This result was obtained during the proof of Theorem 5.3 of [DGH] by using the decomposition polynomial. $\square$

Proposition 3.7 *Every involution in $G_{\mathcal{D}}$ is of type $2B$, so $G_{\mathcal{D}}$ is $2B$-pure.*

Remark 3.8 For a general FVOA, one may not expect the group $G_{\mathcal{D}}$ to be pure.

Proof. From Proposition 3.6, the code $\mathcal{D}$ has one codeword of weight 0, $2^k - 2$ codewords of weight 8 and one of weight 16. For the components V^I of $V = \bigoplus_{I\in\mathcal{D}} V^I$ we have the following decomposition into T_{16}-modules (see again [DGH] proof of Th. 5.3): For $I = 0^{16}$: $V^0 = \bigoplus_{c\in\mathcal{C}} V(c)$ with $\mathcal{C} = \mathcal{D}^{\perp}$. For the weight of I equal to 8: $V^I = \bigoplus_{h_i\in\{0,\frac{1}{2},\frac{1}{16}\}} 2^{5-k} M(h_1, \ldots, h_{16})$, where $h_i = \frac{1}{16}$ for $i \in I$ and $h_i = \frac{1}{2}$ for an odd number of $i \notin I$ and $h_i = 0$ otherwise. For $I = 1^{16}$: $V^I = 2^{8-k} M(\frac{1}{16}, \ldots, \frac{1}{16})$. This gives $8 \cdot (2^{5-k} - 1)$, $8 \cdot 2^{5-k}$, and 2^{8-k} for the dimension of the weight one part V_1^I of V^I, respectively.

Let $\mu \in G_{\mathcal{D}} \cong \widehat{\mathcal{D}}$ be an involution. We investigate its action on the weight one part V_1 of V. The 2^{k-1} codewords $I \in \mathcal{D}$ for which μ acts by -1 on V^I, i.e., with $\mu(I) \neq 0$, have weight 8 or 16. It follows from above that the -1-eigenspace of μ on V_1 has dimension $2^{k-1} \cdot 2^{8-k} = 128$ and the $+1$-eigenspace has dimension $248 - 128 = 120$. Therefore, μ is an involution of type $2B$ described in Prop. 3.2.

$\square$

In our case, the $2B$-pureness of $G_{\mathcal{D}}$ implies that when it is toral (i.e., $k \leq 4$), it lies in a maximal totally singular subspace of $T_{(2)}$. (The natural quadratic form on this subgroup of the torus is given in [G91].) For $E_8(\mathbb{C})$, the set of totally singular subspaces of $T_{(2)}$ of a given dimension form an orbit under the Weyl group. For this reason, $G_{\mathcal{D}}$ is characterized up to conjugacy in $E_8(\mathbb{C})$ by the integer k. See [CG, G91] for a more detailed discussion.

Definition 3.9 Given a frame F, we denote by $t(f)$ the *Miyamoto involution* of type 2 associated to $f \in F$ (see [M96]).

The D-group $G_{\mathcal{D}}(F)$ equals $\langle t(F)\rangle = \langle t(f) \mid f \in F\rangle$, which is naturally identified with $\widehat{\mathcal{D}}$, cf. Prop. 1.1. We have seen that for a frame F the D-group $G_{\mathcal{D}}(F)$ is $2B$-pure of rank k and all values $k \in \{0, 1, 2, 3, 4, 5\}$ are obtained by the frames of [DGH] and described at the beginning of this section. We shall see that the correspondence between frames and $2B$-pure groups is not monic in general. Note also that there is a bijection between D-groups and subVOAs denoted V^0 in [DGH], defined by $\langle t(F)\rangle \mapsto V^{\langle t(F)\rangle}$.

Proposition 3.10 *Let $F = \{\omega_1, \ldots, \omega_{16}\}$ be a Virasoro frame for V and let $k = dim(\mathcal{D}) \in \{1, 2, 3, 4, 5\}$. One has $|\{t(\omega_i) \mid \omega_i \in F\}| = 2^{k-1}$ and so there are 2^{k-1} distinct Miyamoto involutions coming from the frame. Their set-theoretic complement in the D-group $G_{\mathcal{D}}$ is a codimension one subspace.*

Proof. Every Miyamoto involution $t(\omega_i)$ acts on V^I, $I \in \mathcal{D}$, by -1 or 1 depending on whether the index i is contained in the support of I or not. Using the base of the code $\mathcal{D}$ given in Proposition 3.6, we see that $t(\omega_i)$ always acts as -1 on the first base vector and that any combination of signs on the remaining $k-1$ base vectors is possible. $\square$

Corollary 3.11 *The normalizer of the set of Miyamoto involutions induces $AGL(k-1,2)$ on both the D-group and on the set of Miyamoto involutions in it.*

Proof. Let $G_{\mathcal{D}}$ be the D-group. Since $G_{\mathcal{D}}$ is $2B$-pure, $N(G_{\mathcal{D}})/C(G_{\mathcal{D}}) \cong GL(k,2)$ for all $k \in \{1, 2, 3, 4, 5\}$ by Prop. 3.5, whence the Corollary. $\square$

Lemma 3.12 *For all k, the factor group $G/G_{\mathcal{C}}$ embeds in $Sym_d \wr AGL(k-1,2)$, where $d = 2^{5-k}$.*

Proof. By Theorem 2.8 (3) of [DGH], $G/G_{\mathcal{C}}$ must be a subgroup of $Aut(m_h(V)) \leq Aut(\mathcal{D}) \leq Sym_{16}$ (see [DGH], Def. 2.7 for the notation). Using Proposition 3.6 and Theorem C.3 (ii) of [DGH] we see immediately that $Aut(\mathcal{D}) \cong Sym_d \wr AGL(k-1,2)$. $\square$

Remark 3.13 We shall see in Corollary 3.19 that $G(F)$ induces the permutation group $Sym_{2^{5-k}} \wr AGL(k-1,2)$ on F. Unfortunately, this statement does not seem to follow in an obvious way from the previous result.

Definition 3.14 Suppose that the group J acts on the set Ω. A *block* is a nonempty subset $B \subseteq \Omega$ so that $g(B) = B$ or $B \cap g(B) = \emptyset$ for all $g \in J$. A partition of Ω into blocks is called a *system of imprimitivity.* If the only systems of imprimitivity are the trivial ones ($\{\Omega\}$, and all 1-sets), we call the action of J on Ω *primitive.* Otherwise, we call the action *imprimitive.* Note that primitivity implies transitivity if $|\Omega| > 2$.

Theorem 3.15 (Jordan, 1873) *A primitive subgroup of Sym_n which contains a p-cycle, for a prime number $p \leq n-3$, is Alt_n or Sym_n.*

Proof. [W], Th. 13.9. $\square$

Notation 3.16 We say that the partition of the natural number n has *type* or *partition type* $p^a q^b \cdots$ (for distinct natural numbers p, q, $\ldots$) if it has exactly a parts of size p, b parts of size q etc., and the sum $ap+bq+\cdots$ is n. In Sym_n, the stabilizer of such a partition is isomorphic to $Sym_p \wr Sym_a \times Sym_q \wr Sym_b \times \cdots$, and when we write such an isomorphism type for a subgroup of Sym_n, it is understood to be the stabilizer of a partition (unless stated otherwise).

Lemma 3.17 *If $p > 1$, $a > 1$, $Sym_p \wr Sym_a$ (the natural subgroup of Sym_{pa} fixing a partition of type p^a) is a maximal subgroup of Sym_{pa}.*

Proof. Let P be the partition fixed by the natural subgroup $H \cong Sym_p \wr Sym_a$ of Sym_{pa} and let K be a subgroup, $H \le K \le Sym_{pa}$. If K acts primitively on the pa letters, it is Sym_{pa}, by the Jordan theorem 3.15. We suppose K acts imprimitively and seek a contradiction. Let B be a block from such a system of imprimitively and suppose that $B \cap A \neq \emptyset$ for some $A \in P$. Then, since the action of $Stab_H(A)$ on A is Sym_A, we get $A \subseteq B$. Therefore, B is a union of parts of P. Since the action of H on the parts of P is Sym_P, we deduce that $B = A$ since $|B| < pa$. Therefore, P is the system fixed by K, whence $H = K$, and we are done. □

Lemma 3.18 *Let $n = pqr$ with $p > 1$, $q > 1$, $r \ge 1$ and let $H \cong Sym_p \wr Sym_{qr}$ be the subgroup of Sym_n fixing a partition P of type p^{qr}. Let Q be a partition of type $(pq)^r$ of which P is a refinement and let H_0 be the subgroup of Sym_n which permutes the parts of P and fixes all the parts of Q. Thus, $H_0 \cong \prod_1^r Sym_p \wr Sym_q$.*

Let $G_0 \cong \prod_1^r Sym_{pq}$ be the subgroup of Sym_n fixing all parts of Q. Let $g \in G_0$ and write $g = g_1 \cdots g_r$ as a product of permutations for which $supp(g_i)$ is contained in Q_i, the i^{th} part of Q. Suppose that each g_i is not in H_0. Then $\langle H_0, g\rangle = \langle H_0, g_1, \ldots, g_r\rangle = G_0 \cong \prod_1^r Sym_{pq}$.

Proof. Define $J = \langle H_0, g\rangle$ and let $G_i \cong Sym_{pq}$ be the set of permutations which are the identity outside Q_i, the i^{th} part of Q, and let $H_i = H \cap G_i \cong Sym_p \wr Sym_q$.

We study the action of J on Q_i, and use the property that H_i is maximal in G_i (Lemma 3.17). We have the natural projection maps $G_0 = G_1 \times \cdots \times G_r \to G_i$. Since J projects onto G_i (by maximality of H_i and the hypotheses on the g_i), it follows that $J \cap G_i$ is a normal subgroup of G_i. The only normal subgroup of G_i which contains H_i is G_i: Since H_i contains a transposition, a normal overgroup contains all transpositions, so equals G_i. We conclude that J contains each G_i and is therefore equal to G_0. □

Corollary 3.19 *For $k = 1, 2, 3, 4$, let $d = 2^{5-k}$. Then $G/G_{\mathcal{C}} \cong Sym_d \wr AGL(k-1, 2)$.*

Proof. Use Lemma 3.12 and Theorem 3.1. For $k = 4$ we are already done. For $k = 1$, 2, and 3 apply Lemma 3.18 with $p = 2$, $q = 2^{4-k}$, $r = 2^{k-1}$ and use the description of the image $\bar{\sigma} \in G/G_{\mathcal{C}}$ of the triality automorphism σ given in Theorem 2.19. □

Remark 3.20 The conclusion of Cor. 3.19 is also true for $k = 5$, but there is no Cartan subalgebra naturally associated to the frame in the nontwisted lattice construction of V, hence no obvious analogue of Theorem 2.8.

4 The five classes of frames in V_{E_8}

In this section, we describe and characterize the frame stabilizer as a subgroup of $Aut(V) = E_8(\mathbb{C})$. We also prove the uniqueness of the Virasoro frame for $k = 5$.

4.1 The case $k = 1$

Theorem 4.1 (Characterization of G, $k = 1$) *The C-group $G_{\mathcal{C}}$ is extraspecial of plus type, i.e., $G_{\mathcal{C}} \cong 2_+^{1+14}$. Also, $G_{\mathcal{C}}$ is the unique subgroup of its isomorphism type up to conjugacy in $Aut(V)$ and $G = N_{Aut(V)}(G_{\mathcal{C}}) \cong 2_+^{1+14}.Sym_{16}$*

Proof. We now show that $G_{\mathcal{C}}$ is extraspecial. Since $G_{\mathcal{C}}$ is nonabelian and since $G_{\mathcal{D}} \cong 2$ and $G_{\mathcal{C}}/G_{\mathcal{D}}$ is elementary abelian, it follows that $G'_{\mathcal{C}} = G_{\mathcal{D}}$.

Now let Z be the center of $G_{\mathcal{C}}$. It is clear from the action of $O_2(\widetilde{W}_X)$ that $Z \cap G_{\mathcal{C}}$ has order 2, corresponding to the coset $e_i + L$. Due to this action, which is faithful (note that an element corresponding to -1 inverts elements of order 4 in $T \cap G_{\mathcal{C}}$), it follows that Z has order 2. Since $G_{\mathcal{C}}$ has nilpotence class at most 2, it follows that $Z = G'_{\mathcal{C}}$ and so $G_{\mathcal{C}}$ is extraspecial.

From the action of a Sym_8 subgroup of $\widetilde{W}_X$, we see that its composition factors within $G_{\mathcal{C}}$ are irreducibles of dimensions 1 (three times) and 6 (twice), and that the section $G_{\mathcal{C}} \cap T/Z$ is an indecomposable module with ascending factors of dimensions 6, 1 and that $G_{\mathcal{C}}/G_{\mathcal{C}} \cap T$ is the dual module, an indecomposable module with ascending factors of dimensions 1, 6.

By [G91], (12.2), this extraspecial group is uniquely determined up to conjugation in $C_{E_8(\mathbb{C})}(Z) \cong HSpin(16, \mathbb{C})$, and so its normalizer looks like $2_+^{1+14}.Sym_{16}$. By Corollary 3.19, $G = N_{Aut(V)}(G_{\mathcal{C}})$ □

4.2 The case $k = 2$

We get the structure $2^{2+12}[Sym_8 \wr 2]$ for G, Cor. 3.19, but now we want to understand G as a subgroup of $E_8(\mathbb{C})$.

Definition 4.2 A *wreathing element* in a wreath product $G = H \wr 2$ is an element of G outside the direct product $H_1 \times H_2$ of two copies of H which are used in the definition of the wreath product. The same term applies to elements of G/Z outside $H_1 \times H_2/Z$, where Z is a normal subgroup of G such that $Z \cap H_1 = 1 = Z \cap H_2$ (it follows that Z is central).

Theorem 4.3 (Characterization of G, $k = 2$) *G satisfies the hypotheses of this conjugacy result:*
In $E_8(\mathbb{C})$, there is one conjugacy class of subgroups, each of which is a semidirect

product $X\langle t\rangle$, where t has order 2, $X = X_1X_2$ is a central product of groups of the form $[2\times 2^{1+6}]Sym_8$ such that $X_1\cap X_2 = Z(X_1) = Z(X_2)$ and t interchanges X_1 and X_2.

Proof. Let $H := N(G_{\mathcal{D}}) \cong 2^2D_4^2[2 \times Sym_3]$ and let H_1, H_2 be the two central factors of H^0. Define $G_i := G \cap H_1$. From the structure of H/H^0, it is clear that $G_0 := G\cap H^0$ has index dividing 4 in $2^{2+12}[Sym_8 \wr 2]$, whence $G_0 \geq O_2(G)$. For $i \in \{1, 2\}$, let i' be the other index.

First we argue that $G_i \cong [2 \times 2^{1+6}]Sym_8$. Consider the quotient map $\pi_i : H^0 \to H^0/G_i \cong PSO(8,\mathbb{C})$. Then $\pi_i(O_2(G))$ is an elementary abelian 2-group in H^0/G_i, so has rank at most 8 [G91]. Since $G_{\mathcal{D}} \leq ker(\pi_i) = H_{i'}$ and $rank(G_{\mathcal{C}}/G_{\mathcal{D}}) = 12$, it follows that $G_{\mathcal{C}}\cap H_i > G_{\mathcal{D}}$, for all i. Since $G_{\mathcal{C}}\cap H_i$ is normal in G_0, it has order 2^8 or 2^{14}. The latter is impossible since then we would get a rank 14 elementary abelian subgroup of $PSO(8,\mathbb{C})$, which is impossible [G91]. The normal structure of G_0 now implies that $G_i/G_{\mathcal{D}}$ contains a copy of $2^6{:}Alt_8$ and equals this subgroup or has the shape $2^6{:}Sym_8$. We now argue that the latter is the case, and this follows from embedding of G in $C \cong HSpin(16,\mathbb{C})$, the centralizer in $E_8(\mathbb{C})$ of $Z(G) \cong 2$ and using the 16-dimensional projective representation of C, noting that $G/G_{\mathcal{C}}$ maps isomorphically to a natural subgroup of the form $PSO(8,\mathbb{C}) \wr 2$, with image $[2^6{:}Sym_8] \wr 2$. It is clear from this representation, that $G_0 = G_1G_2$ and that the outer elements $G \setminus G_0$ correspond to wreathing elements in the above $PSO(8,\mathbb{C})\wr 2$. In $H/H^0 \cong 2\times Sym_3$ (with the second factor corresponding to simultaneous graph automorphisms on both H_i), such elements would correspond to an involution which projects nontrivially on the first factor. The reason is that a subgroup of $PSO(8,\mathbb{C}){:}Sym_3$ isomorphic to $2^6{:}Sym_8$ is self-normalizing. The wreathing elements in G act nontrivially on $G_{\mathcal{D}}$ (see Corollary 3.11). This means that an involution of $G \setminus G_0$ projects nontrivially on the second factor of $H/H^0 \cong 2 \times Sym_3$. This implies that G is unique up to conjugacy in $E_8(\mathbb{C})$ and this uniqueness follows from just the isomorphism type of G and the fact that $G_{\mathcal{D}}$ is $2B$-pure.

It follows that $|G : G_0| = 2$ and that $G_0 = G_1G_2$ (central product), $G_1\cap G_2 = Z(G_1) = Z(G_2) \cong 2^2$. The isomorphism type of G will be uniquely determined if we show that there is an involution in $G \setminus G_0$. But this follows from the above representation in $SO(16,\mathbb{C})$ since a wreathing involution has spectrum $\{1^8, -1^8\}$ and so lifts in the spin group to an involution [G91, Ch]. For $i = 1$, 2, define z_i by $\langle z_i\rangle := O_2(G_i)' \cong \mathbb{Z}_2$.

Finally, we observe from the above 8-dimensional representations of G_i that $O_2(G_i)/O_2(G_i)' \cong 2^7$ is an indecomposable module for $G_i/O_2(G_i)$ with ascending socle factors of dimensions 1 and 6. Also, for both i, we may think of $O_2(G_i)' \cong 2$ as the kernel of the representation of H_i as $SO(8,\mathbb{C})$ (rather than as a half spin group), since the above wreathing involution is realized in the degree 16 orthogonal projective representation of $C(Z(G))$. It follows that $O_2(G_1)' = O_2(G_2)'$, i.e., $z_1 = z_2$. □

4.3 The case $k = 3$

Let $H := N(G_{\mathcal{D}}) \cong 2^4 A_1^8.AGL(3,2)$; then $H^0 = C(Z(H^0))$ and $C(O_2(H)) \cong H^0.2^3$. See [CG, G91].

We have already determined in Theorem 3.1 that $G_{\mathcal{C}}$ has the shape $2^{3+9} = 2^{4+8}$, the latter decomposition coming from the general structure of H, above.

From earlier sections, we know $G \cap N \cong 2^{3+9}$. This group, with the triality of Theorem 2.19, will generate G, a group of the form below (see Cor. 3.19):

$$[2^{3+9} = 2^{4+8}]Sym_4^4.Sym_4 \cong [2^{4+16}]Sym_3 \wr Sym_4.$$

What we need is a group theoretic characterization of such a subgroup of $E_8(\mathbb{C})$.

Definition 4.4 Let the groups $X_1, \ldots, X_r$ be given and let $X = X_1 \cdots X_r$ be a central product. Let J be the set of indices $\{1, \ldots, r\}$ and let S be a subset of X. Let Z be the subgroup of $Z(X)$ generated by all $X_i \cap \langle X_j \mid j \neq i \rangle$; then X/Z is a direct product of the $X_i Z/Z$. We define the *quasiprojection* of S to $X_J := \langle X_i \mid i \in J \rangle$ to be $X_J \cap S^*$, where S^* is the preimage in X_J of the projection of SZ/Z to X_J/Z, a direct factor of X/Z, complemented by $X_{\{1,\ldots,r\} \setminus J}$. (We use this concept in the case where X_i is quasisimple.)

Let Q_J be the quasiprojection of Q to H_J, $J \subseteq \{1, \ldots, 8\}$.

Theorem 4.5 (Characterization of G, $k = 3$) *Let X be a finite subgroup of $E_8(\mathbb{C})$ such that*

(i) X has the form $[2^{3+9} = 2^{4+8}]Sym_4^4.Sym_4 \cong [2^{4+16}]Sym_3 \wr Sym_4$;

(ii) X has a normal subgroup $E \cong 2^3$ which is $2B$-pure.

Then:

(a) *X is unique up to conjugacy in $E_8(\mathbb{C})$.*

(b) *If $H := N_G(E) \cong 2^4 A_1^8.AGL(3,2)$, then $X \cap H^0 \cong [2^{3+9} = 2^{4+8}]Sym_4^4 = 2^{4+16} Sym_3^4$.*

(c) *If $H_1, \ldots, H_8$ are the $SL(2,\mathbb{C})$-components of H^0, then $H_i \cap X \cong Quat_8$ and there is a partition Π of $\{1, \ldots, 8\}$ into 2-sets so that if $A \in \Pi$, then $\langle H_i \mid i \in A \rangle \cap X \cong [Quat_8 \times Quat_8].Sym_3$ (the top Sym_3 layer sits diagonally over the two $Quat_8$-factors).*

Proof. Denote by U the normal subgroup indicated by $2^{3+9} = 2^{4+8}$. Let $P \in Syl_3(X)$, $P \cong 3 \wr 3 \times 3$. Then, $R := P \cap H^0$ is abelian (property of a connected group of type A_1^8), whence $R \cong 3^4$ is the unique maximal abelian subgroup of P. In H, any 2-group S satisfying $S = [S, R]$ lies in a central

product $Q := Q_1 \cdots Q_8$ of groups $Q_i \cong Quat_8$, $Q_i \leq H_i$, for $i \in \{1, \ldots, 8\}$. It follows that $G_{\mathcal{C}}$ lies in $Q \cong 2^{4+8}$ (see [G91]).

Now, let $Z := O_2(H)$. Then, $Z = Z(Q) = Q'$ and $Q/E \cong 2_+^{1+16}$ is extraspecial with center Z/E. The outer automorphism group of Q/E is $O^+(16,2)$, in which the stabilizer of a maximal isotropic subspace I of the natural module $M := \mathbb{F}_2^8$ has the form

$$2^{120}{:}GL(8,2).$$

In this stabilizer, an elementary abelian subgroup of order 3^4 acts on I by a direct sum of four distinct 2-dimensional irreducibles, and it has equivalent (= dual) action on M/I. It follows that R decomposes into a direct product $R = R_1 \times R_2 \times R_3 \times R_4$ and that the index set $\{1, \ldots, 8\}$ has a partition into 2-sets $A(1)$, ..., $A(4)$ so that $R_i \cong 3$ centralizes H_j iff $j \notin A(i)$ and $[Q, R_i] = Q_{A(i)} \cong 2^{2+4}$ (from [G91], if the involution z_i generates $Z(Q_i)$ and a product of $r > 0$ distinct z_i equals 1, then $r \geq 4$).

So far, we have shown that the conjugacy class of a subgroup of X of the shape $2^{4+16}{:}3^4$ is unique, namely the conjugacy class of the subgroup QR. Our frame stabilizer contains a group Y of the form $2^{4+16}.Sym_3^4 \cong 2^{4+8}.Sym_4^4$ (Cor. 3.19). In $N_G(Q) \cong 2^{4+16}Sym_3^8.AGL(3,2)$, there is a unique group containing QR which has the form $2^{4+16}.Sym_3^4 \cong 2^{4+8}.Sym_4^4$ (so we now have $G \cap H^0$ up to conjugacy). Such a group is contained in H^0 and in fact it is QRT, where $T = T_1 \times \cdots \times T_4$, where $T_i \leq H_{A(i)}$, $T_i \cong 4$ and $R_iT_i \cong 3{:}4$ embeds diagonally in $H_{A(i)} \cong SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. Given the partition, all such choices of T_i are equivalent under conjugation by an element of R^*, the unique group of order 3^8 in the torus $C(R)^0$. We now deduce the uniqueness up to conjugacy of a subgroup of the form $[2^{4+16}.Sym_3^4 \cong 2^{4+8}.Sym_4^4].Sym_4$ by a Frattini argument on the group $T_1T_2T_3T_4Q/Q$ of order 2^4 in $QRT/Q \leq Y/Q$, which maps to H/H^0 as the subgroup of the degree 8 permutation group $AGL(3,2)$ which stabilizes a partition [DGH], Appendix A., p. 441. Therefore, G is determined up to conjugacy. □

Definition 4.6 We use the notation in the proof of the preceeding result. We say that an element of Q may be written as a product of elements in the Q_i and the number of such elements which are outside $Z(Q_i)$ is the Q-*weight*. We define the $Z(Q)$-*weight* of a product of a set of n distinct z_i to be n.

Remark 4.7 In this sense, the products of the z_i which are the identity are those whose support forms a word in a Hamming code H_8 with parameters $[8,4,4]$.

Corollary 4.8 *For $k = 3$, $G'_{\mathcal{C}} = G_{\mathcal{D}}$.*

Proof. It is clear that every element of $G_{\mathcal{C}}/Z(Q)$ has even Q-weight [GR94]. In fact, weights 0, 2, 4 and 8 all occur. It is also clear by looking in the $Q_{A(i)}$ that we get all even words in the generators z_j as commutators in $G_{\mathcal{C}}$. □

4.4 The case $k = 4$

We have $G_{\mathcal{C}} \leq N = N(T)$, Cor. 2.3, and we get $G_{\mathcal{C}} \cap T \cong 4^4$. Also, $G_{\mathcal{C}} \cong 4^4{:}2$ (the outer 2 corresponds to an inverting element (note that every inverting element is an involution in this case [CG, G91]); here, $D_X = \langle -1 \rangle$), $G_{\mathcal{D}} = G'_{\mathcal{C}} = Z(G_{\mathcal{C}})$, $G_{\mathcal{C}}/G_{\mathcal{D}} \cong 2^5$.

Next, $(G \cap T)G_{\mathcal{C}} \cong [2^4 \times 8^4]{:}2$ and $(G \cap T)G_{\mathcal{C}} \triangleleft G \cap N$. Also, $D_X \cong 2, W_X \cong 2.AGL(3,2)$.

So, $G_{\mathcal{D}} \leq O_2(G)'$. Since $C(G_{\mathcal{D}})^0 = T$, we get $G \leq N(G_{\mathcal{D}}) \leq N(T) = N$ and so $G \leq N$. (This is a consequence of Cor. 3.19, proved by different arguments. We remark that since $G = G \cap N$, there is no "triality" in this case, a consequence of the earlier result Cor. 3.19).

The structure of G is given by Theorem 2.8 and so $G \cong [2^4 \times 8^4].2.AGL(3,2) \cong 2^{4+5+8}.AGL(3,2)$.

We now develop a characterization of G as a subgroup of $Aut(V)$.

Definition 4.9 A subgroup $A \cong 4^4$ of $E_8(\mathbb{C})$ is called a $GL(4,2)$-*signalizer* if $N(A)/C(A)$ has a composition factor isomorphic to $GL(4,2)$ (notice that $Aut(A)$ has the form $2^{16}.GL(4,2)$. (This terminology is adapted from the signalizer concept in finite group theory.)

Lemma 4.10 *There is one conjugacy class of $GL(4,2)$-signalizers in $E_8(\mathbb{C})$.*

Proof. Clearly, $N(A)$ has one orbit on the involutions of $E := \Omega_1(A)$. Since there is no pure $2A$-subgroup of rank greater than 3 by [CG], this must be a $2B$-pure group of order 16. Such a group is unique up to conjugacy. Let T be its connected centralizer, a rank 8 torus. In $W := W_{E_8}$, there is no subgroup isomorphic to 2^4 whose normalizer induces $GL(4,2)$ on it, so $A \leq T$. Define $W_A := Stab_W(A)$ and $W_E := Stab_W(E)$. Then $W_E \cong 2^{1+6}GL(4,2)$ and W_A is a subgroup of this containing a composition factor isomorphic to $GL(4,2)$. It is an exercise with the Lie ring technique that $Aut(A)$ does not contain $GL(4,2)$ [G76]. We now show that $W_A < W_E$. If false, $O_2(W_E)$ acts trivially on E and A/E, so acts as an elementary abelian 2-group, whereas the element of W corresponding to -1 must act nontrivially, yet is in the Frattini subgroup of $O_2(W_E)$, contradiction. Since $O_2(W_E)$ has just two chief factors under the action of W_E, the only possibility for $W_A < W_E$ is $W_A \cong 2{\cdot}GL(4,2)$, the nonsplit extension.

Let $U := T_{(2)}$. Then W_A normalizes $UA = \langle U, A \rangle \cong 2^4 \times 4^4$. So does W_E, since U is the inverse image in $T_{(2)}$ of E under the squaring endomorphism. The action of W_E on $UA/E \cong 2^8$ has kernel exactly $Z(W_E) \cong 2$ (otherwise, we would have $[A, O_2(W_E)] \leq [U, O_2(W_E)] \leq E$ and get a contradiction as in the previous paragraph). It follows that in UA/E, the subspace A/E has stabilizer in W_E equal to W_A. Also, UA/E is a completely reducible W_A-module, the direct sum of a four dimensional module and its dual.

Now let A_1 be another $GL(4,2)$ signalizer in $E_8(\mathbb{C})$. By following the above procedure for A_1 in place of A, we may arrange $UA_1 = UA$. We may replace A_1 by a conjugate with an element of W_E so that both A and A_1 are stabilized by W_A (property of a parabolic subgroup of $O^+(8,2)$). Since there are precisely three W_A-chief factors within UA, two isomorphic to the natural module for $W_A/Z(W_A) \cong GL(4,2)$ and the third to the dual module, it follows that $A \cap A_1 \geq E$ and finally that $A = A_1$. $\square$

Theorem 4.11 (Characterization of G, $k = 4$) *The group G is determined up to conjugacy, as described below, in the normalizer of A, a $GL(4,2)$-signalizer, which is, in turn, unique up to conjugacy in $E_8(\mathbb{C})$.*

The normalizer in $E_8(\mathbb{C})$ of A, a $GL(4,2)$-signalizer is a group of the form $T.[2{\cdot}GL(4,2)]$. In $N(A)$, let u be any element which acts on T by inversion. Then $|u| = 2$. Set $B := \langle A, u\rangle$. Then $N(B)$ has the form $4^4.2^8.2{\cdot}GL(4,2)$ and the normal subgroup of the shape $4^4.2^8 \cong 2^4 \times 8^4$ is a characteristic subgroup of $O_2(B)$. The group G is a subgroup of index 15 in $N(B)$ stabilizing a rank three subgroup of $\Omega_1(A)$, and this property characterizes G up to conjugacy in $N(B)$.

Proof. The maximal subgroup above of $O_2(B)$ is the unique maximal subgroup which is abelian, so is characteristic. $\square$

4.5 The case $k = 5$

Let F be a frame with $k = 5$. The D-group $G_{\mathcal{D}}$ is the unique up to conjugacy $2B$-subgroup of rank 5 in $Aut(V)$ and the normalizer of $G_{\mathcal{D}}$ is $2^{5+10}GL(5,2)$ (see Prop. 3.5). More precisely, one has:

Proposition 4.12 *The extension*

$$1 \to O_2(N(G_{\mathcal{D}}))/Z(O_2(N(G_{\mathcal{D}}))) \to N(G_{\mathcal{D}})/Z(O_2(N(G_{\mathcal{D}}))) \to GL(5,2) \to 1$$

is split, though $N(G_{\mathcal{D}})$ does not split over $O_2(N(G_{\mathcal{D}}))$. There is a subgroup X of $N(G_{\mathcal{D}})$ satisfying $Z(O_2(N(G_{\mathcal{D}}))) \leq X$, $X/Z(O_2(N(G_{\mathcal{D}}))) \cong GL(5,2)$ is nonsplit over $Z(O_2(N(G_{\mathcal{D}})))$.

Proof. [G76], Section 1. $\square$

The group X has been called *the Dempwolff group*, [Th, G76], and we call $N(G_{\mathcal{D}})$ the *Alekseevski-Thompson group*. The first discussion of this group in the literature on finite subgroups of Lie groups was probably [A], but Thompson discovered it independently around early 1974 while studying the sporadic group F_3 which embeds in $E_8(3)$ [Th, G76].

It follows from Corollary 3.11 that the frame stabilizer is contained in the subgroup of $N(G_{\mathcal{D}})$ which preserves the set $t(F)$ of Miyamoto involutions, an affine hyperplane of $G_{\mathcal{D}}$.

Though t restricted to F gives a bijection with a set of 16 involutions in $G_{\mathcal{D}}$, it is possible that some other frame $F' \neq F$ satisfies $t(F) = t(F')$. We shall prove now that an affine hyperplane $t(F) \subset G_{\mathcal{D}} \leq E_8(\mathbb{C})$ corresponds to a unique frame in V, i.e., all VFs with $k = 5$ are equivalent under $Aut(V)$, so in particular are equivalent to the VF Ω of [DGH].

Let $V^0 = V^{G_{\mathcal{D}}}$ be the subVOA fixed by the D-group. It is the vertex operator algebra studied in [G98]. It has automorphism group $O^+(10,2)$, graded dimension $1 + 0\,q^1 + 156\,q^2 + \cdots$ and is isomorphic to the VOA $V^+_{\sqrt{2}E_8}$.

Proposition 4.13 *The action of the normalizer $N(G_{\mathcal{D}}) \cong 2^{5+10}GL(5,2)$ induces the action of a parabolic subgroup $P \cong 2^{10}{:}GL(5,2)$ of $O^+(10,2)$ on V^0.*

Proof. The kernel of the action of G on V^0 is $G_{\mathcal{D}}$. In fact, if K is the (possibly larger) subgroup of $E_8(\mathbb{C})$ which acts trivially on V^0, then it acts trivially on the frame F, so is contained in G. Therefore, $K = G_{\mathcal{D}}$.

Thus $N(G_{\mathcal{D}})/G_{\mathcal{D}}$ acts faithfully on V^0, so gives a subgroup of shape $2^{10}{:}GL(5,2)$ of $O^+(10,2)$. Such a subgroup is unique up to conjugacy. It is the stabilizer of a maximal totally singular subspace in $\mathbb{F}_2^{10}$. □

The relevant Virasoro elements of V^0 are 496 in number and may be identified with the nonsingular points $\mathcal{N}$ in the space $\mathbb{F}_2^{10}$ with a maximal Witt index quadratic form (see [G98], 6.8). Under this identification, a Virasoro frame of V^0 corresponds to a set of nonsingular vectors in $\mathbb{F}_2^{10}$ spanning a 5-dimensional subspace which is totally singular with respect to the bilinear form associated to the quadratic form (reason: all Miyamoto involutions σ of type 2 associated to a frame commute and commutativity corresponds in this case to orthogonality of the corresponding members of $\mathcal{N}$). In it, the set of nonsingular vectors is the nontrivial coset of the 4-dimensional subspace of totally singular vectors and zero.

Recent work of Lam [L] shows that $Aut(V^0)$ is transitive on frames within V^0. One should keep in mind that frames in V^0 may have different values of k as frames in V. Next, we give a short proof of transitivity on frames.

Proposition 4.14 *In V^0, there is one orbit of $Aut(V^0)$ on frames.*

Proof. By Witt's theorem, the 5-dimensional subspace corresponding to a frame is unique up to isometry of $\mathbb{F}_2^{10}$. Transitivity of $Aut(V^0)$ on its frames follows at once. □

The orbits of the parabolic subgroup P on such subspaces are studied in Appendix 5.4. We use this to show:

Theorem 4.15 *For $k = 5$, there is one $E_8(\mathbb{C})$ orbit of such Virasoro frames in V. If F, F' are frames with $k = 5$ and $t(F) = t(F')$, then $F = F'$.*

For its proof we need:

Lemma 4.16 *For $k \leq 4$, $G_{\mathcal{C}}$ contains a group E^* which is elementary abelian of order 2^5 and is $2B$-pure, contains $G_{\mathcal{D}}$ and has the property that $T \cap E^*$ has rank 4.*

Proof. Let u be any element of $G_{\mathcal{C}}$ which corresponds in N/T to -1. Then u is an involution (see [G91]) and it centralizes $T_{(2)}$. For all $k \leq 4$, such u exist in $G_{\mathcal{C}}$ and $G_{\mathcal{C}}$ contains a maximal totally singular subspace of $T_{(2)}$ containing $G_{\mathcal{D}}$, say E_1. Take $E^* = E_1\langle u \rangle$. This group is $2B$-pure, see [G91]. □

Proof of Theorem 4.15. The group $G_{\mathcal{D}}$ of a VF with $k = 5$ is up to conjugation unique in $E_8(\mathbb{C})$ and has normalizer $2^{5+10}GL(5,2)$ which induces the action of $2^{10}{:}GL(5,2) \cong P$ on V^0 by Prop. 4.13.

We claim that, for every $k \in \{1, 2, 3, 4, 5\}$, there is a frame F' from V with $k = dim(G_{\mathcal{D}}(F'))$ which is contained in V^0, the fixed point VOA for $G_{\mathcal{D}}$. For $k = 5$, this is obvious. For $k \leq 4$, this will follow if we show that the group $G_{\mathcal{C}}(F')$ contains a conjugate of $G_{\mathcal{D}}$, since if g^{-1} is an element conjugating $G_{\mathcal{D}}$ into $G_{\mathcal{C}}(F')$ one has $g(F') \subset V^{gG_{\mathcal{C}}(F')g^{-1}} \subset V^{G_{\mathcal{D}}}$. By Lemma 4.16 and Prop. 3.5, we are done.

On a P-orbit, the values of k are constant. Since Proposition 5.13 shows that we have five orbits and all five values of k are represented there, P acts transitively on the set of frames of V^0 with a fixed value of k, and in particular we have transitivity for $k = 5$. □

We refer to Section 5.4 for the definition of the J-indicator and the collection of subspaces, Σ. We need to know something about the (k, j)-*bijection* $\{1, 2, 3, 4, 5\} \leftrightarrow \{0, 1, 2, 3, 4\}$ indicated in the proof of Th. 4.15.

Definition 4.17 Define P_j as the stabilizer in P of some member of Σ with J-indicator j. Using $G/G_{\mathcal{D}} \cong P$, let H_j be the subgroup of $N(G_{\mathcal{D}})$ with $H_j/G_{\mathcal{D}} \cong P_j$.

Lemma 4.18 *The cases of J-indicator 0 or 4 correspond to frames $F' \subset V^0$ with $k = 1$ or 5. Thus, $\{1, 5\} \leftrightarrow \{0, 4\}$ under the (k, j)-bijection.*

Proof. From Prop. 5.14 and Cor. 5.15, H_j has a normal subgroup Q of order 2^9 resp. 2^{19} and $H_j/Q \cong GL(4, 2)$. Let F' be the associated frame. So, $H_j \leq G(F')$ and we want to show that $k = 1$ or 5 for F'.

We assume $k \neq 1$, 5 and derive a contradiction. According to the shapes of frame stabilizers for $k \in \{2, 3, 4\}$, the fact that $H_j/Q \cong GL(4, 2)$ implies $k = 2$. In H_j, the chief factors afford irreducible modules for $H_j/Q \cong GL(4, 2)$ of dimensions 1, 4 and 6. In a frame stabilizer for $k = 2$, only dimensions 1 and 6 occur. □

Lemma 4.19 *Let K be a 4-dimensional subgroup of $G_{\mathcal{D}} \cong 2^5$. Then, $O_2(N(G_{\mathcal{D}}))/K \cong 2^{5+10}/K$ is isomorphic to the direct product 2^4 with the extraspecial group 2^{1+6}.*

Proof. This follows from (3) in Section 3 of [G76]. The above quotient group is nonabelian with derived group of order 2 and admits the action of $N(G_{\mathcal{D}}) \cap N(K) \cong 2^{5+10}AGL(4,2)$. Since this subgroup embeds in a torus normalizer, the above structure is forced. $\square$

Proposition 4.20 *The frame F with $k = 5$ corresponds to the J-indicator 0. Therefore, $5 \leftrightarrow 0$ and $1 \leftrightarrow 4$ under the (k, j)-bijection.*

Proof. Take $k = 5$. We assume by Lemma 4.18 that the J-indicator is 4 and work for a contradiction. Then by Corollary 5.15, $G(F) = H_4 \cong 2^{5+10}AGL(4,2)$ and $G_{\mathcal{C}} \cong 2^{5+10}$.

The action of $G_{\mathcal{C}}$ on V respects the decomposition into T_{16}-modules. As in the proof of Prop. 3.7 (cf. also [M98]) one has that for an 8-set $I \in \mathcal{D}$ each irreducible T_{16}-module $M(h_1, h_2, \ldots, h_{16})$ with $h_i = \frac{1}{16}$ for $i \in I$ and $h_i = \frac{1}{2}$ for an odd number of $i \notin I$ has multiplicity 1. Let M be one of the eight irreducibles from above with $h_i = \frac{1}{2}$ for exactly one i. In M, the weight 1 subspace $M \cap V_1$ is one dimensional.

The action of $G_{\mathcal{D}} \cong \widehat{\mathcal{D}}$ on $V^I \supset M$ has as kernel the 4-dimensional subgroup K of elements in $\widehat{\mathcal{D}}$ whose kernel contain I. By Lemma 4.19, one has $G_{\mathcal{C}}/K \cong 2^{1+6} \times 2^4$, where the center of the extraspecial group 2^{1+6} is $G_{\mathcal{D}}/K$. Faithful irreducible modules for 2^{1+6} have dimension 8 (cf. [Go, H]), so we have a contradiction.

The Proposition follows now from Lemma 4.18. $\square$

Theorem 4.21 (Characterization of G, $k = 5$) *The frame stabilizer for $k = 5$ has shape $2^5AGL(4,2) \cong 4^4[2 \cdot GL(4,2)]$, where the factor 2 indicates an involution inverting the normal 4^4 subgroup. Such a subgroup is uniquely determined up to conjugacy as a subgroup of $E_8(\mathbb{C})$ by these conditions. In particular, $G_{\mathcal{C}} = G_{\mathcal{D}}$ is elementary abelian.*

Proof. The structure of $G/G_{\mathcal{C}}$ follows from Prop. 4.20 and Cor. 5.15.

For the characterization, one can modify the argument of Theorem 4.11. To do so, we must locate a suitable $GL(4,2)$-signalizer within G, see Def. 4.9.

(i) We define the group A by $A := [O_2(G), G]$. It follows from the vanishing of the Ext^1 group for modules $\{1, 4\}$ (in either order) that $O_2(G) = A\langle t(f)\rangle$, a semidirect product, that the four dimensional chief factors in $G_{\mathcal{C}}$ are isomorphic (by commutation with an element of $t(f)$, for $f \in F$). Since these irreducibles are not self dual, $O_2(G)'$ has rank 4 and $O_2(G) = A\langle t(f)\rangle$ is a semidirect product, for any $f \in F$.

We want to show that A is homocyclic of type 4^4. For $GL(4,2)$, the 6 dimensional module $4 \wedge 4$ is irreducible, so A is abelian. We assume that A is elementary abelian and derive a contradiction.

Let $A \leq E$, a maximal elementary abelian 2-subgroup of $Aut(V) \cong E_8(\mathbb{C})$. The classification in [G91] shows just two conjugacy classes of such E, of ranks 8 and 9. If $rank(E) = 8$, we have a contradiction since $GL(4,2)$ is not involved in $N(E)$. Therefore, $rank(E) = 9$, which means that E contains a unique subgroup E_1 of index 2 which lies in a maximal torus, T (so $E_1 = T_{(2)}$). Such a subgroup is characterized in E as the unique maximal subgroup of E whose complement in E contains only involutions in class $2B$. From [G91], we may take T to be the connected centralizer of $A_1 := O_2(G)'$, a $2B$-pure rank 4 elementary abelian subgroup. In $C(A_1)/T \cong 2^{1+6}$, $O_2(G)$ maps to a subgroup whose normalizer in $N(A_1) \cong T.2^{1+6}GL(4,2)$ has a section isomorphic to $GL(4,2)$. This means $O_2(G) \leq T\langle u\rangle$, where u is an involution in the torus normalizer corresponding to -1 in the Weyl group.

It follows that $A = [O_2(G), G] \leq [T\langle u\rangle, G] \leq T$. Therefore, $A \leq E_1$, so $A = E_1$. We now have a final contradiction, since E_1 is a self dual module for its normalizer, whereas A has chief factors consisting of two non self dual modules, contradiction. This proves $A \cong 4^4$.

(ii) In G, the unique normal abelian subgroup maximal with respect to containment is A. $\square$

5 Appendix

5.1 Equivariant unimodularizations of even lattices

Sometimes it is convenient to have a lattice embedded in another lattice whose determinant avoids certain primes, and it can be useful to do this in a way which respects automorphisms.

First, we recall some basic facts concerning extensions of lattices (cf. [N]). An even lattice L defines a *quadratic space* (A,q), where $A = L^*/L$, $L^* = \{x \in L \otimes \mathbb{Q} \mid (x,y) \in \mathbb{Z} \text{ for all } y \in L\}$ the dual lattice, and $q : L^*/L \longrightarrow \mathbb{Q}/2\mathbb{Z}$ is the quadratic form $x \pmod{L} \mapsto (x,x) \pmod{2\mathbb{Z}}$. Even overlattices M of L define isotropic subspaces $C = M/L$ of (A,q) and this correspondence is one to one. An automorphism g of L extends to an automorphism of M if and only if the induced automorphism $\bar{g} \in Aut(A,q)$ fixes the subspace C. A subgroup C of A generated by a set of elements is isotropic if the generating elements are isotropic and orthogonal to each other with respect to the $\mathbb{Q}/\mathbb{Z}$-valued bilinear form obtained by taking the values of $b(x,y) = \frac{1}{2}(q(x+y) - q(x) - q(y))$. The determinant $det(L)$ of L is the order of A. If A has exponent N, then q takes values in $\frac{1}{2N}(2\mathbb{Z})/2\mathbb{Z}$. There is an orthogonal decomposition $(A,q) = \bigoplus_{p|det(L)}(A_p, q_p)$ of quadratic spaces, where A_p is the Sylow p-subgroup of A.

A sublattice L of a lattice M is called *primitive* if M/L is free. Let $K = L^{\perp}_M$ the orthogonal complement of L in M. Then, L is primitive exactly if the projection of $M/(L \oplus K)$ to K^*/K is injective.

Definition 5.1 Let M be a lattice and L a sublattice. We say that an automorphism α of L *extends (weakly)* to M if there is $\beta \in Aut(M)$ so that $\beta|_L = \alpha$. We say that a subgroup $S \leq Aut(L)$ *extends (weakly)* to M if every element extends and we say that it *extends strongly* if there is a subgroup $R \leq Aut(M)$ which leaves L invariant and the restriction of R to L gives an isomorphism of R onto S. In this case, call such R a *strong extension of S to M.*

Definition 5.2 Let L be an even lattice. An *equivariant unimodularization* of L is an unimodular lattice M containing L as a primitive sublattice such that $Aut(L)$ extends strongly to M.

Theorem 5.3 (James, [J]) *An equivariant unimodularization M of an even lattice L exists of rank at most $2 \cdot rank(L) + 2$.*

One can take for M the orthogonal sum of $rank(L)+1$ hyperbolic planes. A somewhat stronger result can be found in [N] (see Prop. 1.14.1 and Th. 1.14.2).

The above unimodularizations from [J, N] are all indefinite. The next theorem shows that one can get equivariant definite unimodularizations of a definite lattice. For its proof we need:

Lemma 5.4 *(i) Let p be an odd prime and $r \geq 0$. Then, -1 is the sum of two squares in $\mathbb{Z}/p^r\mathbb{Z}$. (ii) For all $r \geq 0$, one can write -1 is a sum of four squares in $\mathbb{Z}/2^r\mathbb{Z}$.*

Proof. (i) When $r = 1$, we quote the well known fact that every element in a finite field is a sum of two squares. Part (i) is now proved by induction on r: Assume that $r \geq 1$ and that a, b are integers such that $a^2 + b^2 = -1 + p^r m$, for some integer m. Let x, y be integers and consider $(a + p^r x)^2 + (b + p^r y)^2 = -1 + p^r m + 2p^r[ax + by] + p^{2r}e$, for some integer e. Since not both a and b can be divisible by p we can solve $2[ax + by] \equiv -m \pmod p$ for integers x, y. Thus, -1 is a sum of two squares modulo p^{r+1}.

Part (ii) follows from a similar argument, or from Lagrange's theorem that every nonnegative integer is a sum of four integer squares. $\square$

Theorem 5.5 *Let L be an even lattice of signature (n_1, n_2). Then there exists an equivariant unimodularization of the lattice L whose rank is $8 \cdot rank(L)$ and signature is $(8n_1, 8n_2)$. If $det(L)$ is odd, there is one whose rank is $4 \cdot rank(L)$ and signature is $(4n_1, 4n_2)$. In particular, if the lattice is definite, this unimodularization is also definite.*

Proof. Assume first that $det(L)$ is odd and let (A, q) be the finite quadratic space associated L. Let $K = L \perp L \perp L \perp L$ having the associated quadratic space $(B, q') = (A, q) \oplus (A, q) \oplus (A, q) \oplus (A, q)$. We decompose (A, q) as the orthogonal sum

$$(A, q) = \bigoplus_{p|det(L)} (A_p, q_p),$$

where $A_p \cong \mathbb{Z}/p^{a_{p,1}}\mathbb{Z} + \mathbb{Z}/p^{a_{p,2}}\mathbb{Z} + \cdots + \mathbb{Z}/p^{a_{p,n_p}}\mathbb{Z}$ is an abelian p-group with $a_{p,1} \geq a_{p,2} \geq \cdots \geq a_{p,n_p}$ of order p^{a_p}, where $a_p := a_{p,1} + a_{p,2} + \cdots + a_{p,n_p}$.

Fix a prime $p|det(L)$. Using Lemma 5.4, let r, $s \in \mathbb{Z}$ so that $r^2 + s^2 \equiv -1 \pmod{p^{a_{p,1}}}$. We let

$$D_p = \{(rx, sx, 0, x) \mid x \in A_p\} \qquad \text{and} \qquad E_p = \{(sx, -rx, x, 0) \mid x \in A_p\}.$$

Since $q_p(\pm rx) + q_p(\pm sx) + q_p(\pm x) = p^{a_{p,1}} q_p(x) \in 2\mathbb{Z}/2\mathbb{Z}$, the groups D_p and E_p are isotropic subspaces of $(A_p, q_p) \oplus (A_p, q_p) \oplus (A_p, q_p) \oplus (A_p, q_p)$. They are orthogonal to each other, so that $C_p = D_p + E_p$ is also isotropic and has order p^{2a_p}.

Finally, let $C = \bigoplus_{p|det(L)} C_p$. It is an isotropic subspace of (B, q') with $|A|^2$ elements and it is invariant under the diagonal action of $Aut(L)$ induced on (B, q'). Since $|C|^2 = |B|$, the overlattice M of K belonging to $C = M/K \leq B$ is a definite even unimodular lattice having an automorphism group which contains a strong extension of $Aut(L)$ and L is also primitive.

Now, we do the case of even $det(L)$. This time we take $K = L \perp \cdots \perp L$ (8 times) with associated quadratic space $(B, q') = (A, q) \oplus \cdots \oplus (A, q)$ (8 times). We proceed in a similar spirit:

For $p = 2$, let r, s, t, u be integers such that $r^2 + s^2 + t^2 + u^2 \equiv -1 \pmod{2^{a_{2,1}+1}}$ and define

$$\begin{array}{rcl} D_2 & = & \{(\ rx, \quad sx, \quad tx, \quad ux, \quad x, \quad 0, \quad 0, \quad 0) \mid x \in A_2\}, \\ E_2 & = & \{(\ sx, \quad -rx, \quad ux, \quad -tx, \quad 0, \quad x, \quad 0, \quad 0) \mid x \in A_2\}, \\ F_2 & = & \{(-x, \quad 0, \quad 0, \quad 0, \quad rx, \quad sx, \quad tx, \quad ux) \mid x \in A_2\} \text{ and} \\ G_2 & = & \{(\ 0, \quad -x, \quad 0, \quad 0, \quad sx, \quad -rx, \quad ux, \quad -tx) \mid x \in A_2\}. \end{array}$$

Since $q_2(\pm rx) + q_2(\pm sx) + q_2(\pm tx) + q_2(\pm ux) + q_2(\pm x) = 2^{a_{2,1}+1} q_2(x) = 2\mathbb{Z}/2\mathbb{Z} \in \mathbb{Q}/2\mathbb{Z}$, the groups D_2, E_2, F_2 and G_2 are totally isotropic subspaces of $(A_2, q_2) \oplus \cdots \oplus (A_2, q_2)$. They are pairwise orthogonal, so that $C_2 := D_2 + E_2 + F_2 + G_2$ is also isotropic and has order 2^{4a_2}.

For the odd primes, we let $C_p = (D_p + E_p) \oplus (D_p + E_p)$. As in the preceding cases, we see that the overlattice M of K belonging to $C = \bigoplus_{p|det(L)} C_p$ has all the required properties. $\square$

If we try only to double the rank of L, we may not find an unimodularization in general, but we can achieve the following:

Theorem 5.6 *Let L be an even lattice with signature (n_1, n_2). Then there exists an even lattice M of signature $(2n_2, 2n_2)$ containing L as a primitive sublattice such that $Aut(L)$ can be strongly extended to a subgroup of $Aut(M)$ and $det(M)$ is a power of an arbitrarily large prime.*

Proof. The Dirichlet Theorem implies that there are infinitely many primes s satisfying $s \equiv -1 \pmod{2\, det(L)}$. Let s be such a prime.

Let $L[s]$ be a lattice which as a group is isomorphic to L by $\psi : L \longrightarrow L[s]$ with bilinear form defined by $(\psi(x), \psi(y)) = s \cdot (x, y)$. Then, $det(L[s]) = s^n det(L)$, where $n = rank(L)$. Extend ψ to maps between the rational vector spaces spanned by L and $L[s]$.

We will define M as an overlattice of $K = L \perp L[s]$. Proceeding as in the proof of the last theorem, let $C = \{(x, \psi(x)) \mid x \in A\} \leq (B, q')$. We have $q'((x, \psi(x))) = (1 + s)q(x) \in 2\mathbb{Z}/2\mathbb{Z}$, i.e., C is isotropic. The determinant of the overlattice M belonging to C is $det(K)/|C|^2 = s^n det(L)^2/det(L)^2 = s^n$. We get an extension of $Aut(L)$ to M by taking the diagonal subgroup of $Aut(L) \times Aut(L[s])$, with respect to the isomorphism ψ. This diagonal subgroup preserves both $L \perp L[s]$ and C hence also M. $\square$

5.2 Lifting $Aut(L)$ to the automorphism group of V_L

This section is written by the first author (RLG).

We need to describe a lifting of the automorphism group W of the even lattice L to a group of automorphisms $\widetilde{W}$ of the VOA V_L. This group will have a normal elementary abelian subgroup of rank equal to the rank of L and the quotient by it will be W. It will normalize a natural torus of automorphisms, whose rank is again equal to the rank of L.

For preciseness, we define the group $\widetilde{W}$ by using the definition of $\hat{L}$, the unique (in the sense of extensions) group such that there is a short exact sequence $1 \to \{\pm 1\} \to \hat{L} \to L \to 1$ with the property that $x^2 = (-1)^{\frac{1}{2}(x,x)}$ (it follows that $[x, y] = (-1)^{(x,y)}$) for x, $y \in L$ (inner products for elements of $\hat{L}$ are evaluated on their images in L).

Definition 5.7 We define $\widetilde{W}$ as the subgroup of the automorphism group of the abstract group $\hat{L}$ which preserves the given quadratic form on L via its action on the quotient by $\{\pm 1\}$. We call $\widetilde{W}$ the *group of isometry automorphisms of* $\hat{L}$.

A construction will exhibit structure of $\widetilde{W}$ and make useful actions available. It is clear from the definition that $\widetilde{W}$ participates in a short exact sequence $1 \to 2^n \to \widetilde{W} \to W \to 1$ where $n = rank(L)$, so that any group constructed as a subgroup of the group of isometry automorphisms which fits into the middle of such a short exact sequence must be the group of isometry automorphisms. So, the choices made in a particular construction do not affect the isomorphism type of the group constructed.

Our construction expresses some unity between two different contexts where such extensions have occurred. In the case where L is between the root lattice and its dual, for a root system Φ whose indecomposable components have types ADE, the group $\widetilde{W}$ will be a subgroup of $G{:}\Gamma$, where G is a simply connected and connected group of type Φ and Γ is the group of Dynkin diagram automorphisms, lifted to G. The basic reference for such a lift is [Ti], which relates the lift to the sign problem for the definition of a Lie algebra, given a root system. When L is the Leech lattice, we get a group $\widetilde{W}$ which comes up in the theory of the Monster simple group [G81, G82]. Other examples of lattices and sporadic simple groups come up this way. See Remark 5.8 for more background.

We now present this author's basic theory of $\widetilde{W}$. It is self-contained, except for applications to VOA theory, for which we assume the standard theory of the VOA structure on the space V_L used in Prop. 5.9. The use of 2-regularizations here is probably new.

Remark 5.8 (Some history) Several finite dimensional representations (including projective ones) of such groups had been known for a long time to group theorists and Lie theorists. When L is the root lattice of a type ADE Lie algebra, $\widetilde{W}$ is a subgroup of the maximal torus normalizer in the simply connected group generated by explicitly defined lifts of the fundamental reflections and the elements of order 2 in the torus [Ti]. In finite group theory, one looks at the study of 2-local subgroups in finite simple groups, especially centralizers of involutions, for occurrences of groups like $\widetilde{W}$, up to central extension. For this author, his involvement started with [G73], in which certain linear groups of shape $2_{\varepsilon}^{1+2n}.O^{\epsilon}(2n,2)$ and $4.2^{2n}.Sp(2n,2)$ were constructed and analyzed. (A few years later, the article [BRW] came to his attention.) See especially the ideas in [G76, G76b], which this author adapted in 1979 to an action of $\widetilde{W}$ on the vector space V_L where L is the Leech lattice, and the same idea worked without change for odd determinant lattices. A version of his ideas was reported in [K], but the report seems to be flawed. See also [G81, G82, G86].

The VOA concept came later, in the mid 80s. See the basic reference [FLM] for VOA theory, which gives a treatment of $\hat{L}$, $\widetilde{W}$ and its action on certain VOAs. The action of $\widetilde{W}$ defined earlier on the vector space V_L turns out to respect the VOA structure on V_L. One can describe $\widetilde{W}$ as the set of group automorphisms of $\hat{L}$ which preserve the bilinear form on L, a neat characterization [FLM] on which the Definition 5.7 is based. The object $\hat{L}$ is not needed to construct $\widetilde{W}$ but was used heavily in [FLM]; possibly these authors were the first to construct $\hat{L}$ and show its relevance to VOA theory. The sign problem for constructing Lie algebras and VOAs had a new solution in the late 70s with the so-called epsilon function [FK, Se] and the epsilon function was later used as a cocycle for creating the group extension $\hat{L}$. See [G96] and references therein for a general discussion about structure constants and group extensions.

5.2.1 The construction of $\widetilde{W}$ when $det(L)$ is odd

For simplicity at first, let us consider the case where L has odd determinant. This is equivalent to the nonsingularity of the $\mathbb{F}_2$-valued bilinear form on $L/2L$ derived from the integer valued one on L by reduction modulo 2. Then n is even and there is an extraspecial 2-group E, unique up to isomorphism, so that the squaring and commutator maps from E to E' are essentially the $\mathbb{F}_2$-valued quadratic form and bilinear form on $L/2L$.

Let M be the essentially unique faithful irreducible module for E. The action of E extends to the faithful action of a group $B \cong E.W$ in such a way that the action of B on E/E' is identified with the action of W on $L/2L$. The group $\widetilde{W}$ is defined by a pullback diagram:

$$\begin{array}{ccc} \widetilde{W} & \longrightarrow & W \\ \downarrow & & \downarrow \\ B/E' & \longrightarrow & W/\{\pm 1\} \end{array}$$

The extension $1 \to E' \to E \to L/2L \to 0$ is given by a cocycle $\varepsilon : L \times L \to \{\pm 1\}$ (and identification of $\{\pm 1\}$ with E'), which is bilinear as a function and so is constant on pairs of cosets of $2L$ in L.

The cocycle ε may be used in an obvious way to construct a group $\hat{L}$, which participates in the short exact sequence $1 \to Z \to \hat{L} \to L \to 0$, where $Z := \langle z \rangle \cong \mathbb{Z}_2$, and which maps onto the group E. Since then $\hat{L}$ also participates in a pullback diagram

$$\begin{array}{ccc} \hat{L} & \longrightarrow & L \\ \downarrow & & \downarrow \\ E & \longrightarrow & L/2L, \end{array}$$

the construction of $\widetilde{W}$ and the above diagram makes it clear that it acts faithfully as a group of automorphisms of $\hat{L}$.

5.2.2 The construction of $\widetilde{W}$ for arbitrary nonzero values of $det(L)$

In general, the lattice L will not have odd determinant, so a modification of the above program is needed. One way is to choose a different (nonabelian) finite 2-group to play the role of E, but the modifications to the previous argument which one seems to need are not attractive. Instead, our idea is to embed L suitably in an odd determinant lattice and deduce what we need for L. This is achieved by any equivariant embedding into a lattice of odd determinant, see Theorems 5.3, 5.5 and 5.6. We call such a lattice a *2-regularization of L.*

We now carry out the earlier construction for J, a 2-regularization of L. We have our definition of $\widetilde{W}_J$, the group of isometry automorphisms of $\hat{J}$.

Since $\hat{L}$ is naturally a subgroup of $\hat{J}$, for which the pullback diagrams are compatible, with compatible epsilon-functions, there is in W_J, by definition of 2-regular extension, a subgroup W_L which we may identify with $Aut(L)$. Let

$\widetilde{W}_{L,J}$ be the preimage of W_L in $\widetilde{W}_J$. Finally, the group $\widetilde{W}$ we seek for $\hat{L}$ is just the image of $\widetilde{W}_{L,J}$ in $Aut(\hat{L})$. The kernel of this map is the normal elementary abelian 2-subgroup of rank equal to $rank(J) - rank(L)$ in $\widetilde{W}_{L,J}$ which acts trivially on $\hat{L}$.

5.2.3 Proof that $\widetilde{W}$ acts faithfully as a group of VOA automorphisms

We shall define the action of $\widetilde{W}$ on V_L below. The definition will make it clear that $\widetilde{W}$ acts as a group of invertible linear transformations on V_L which respects grading.

We have the space $V := V_L = \mathbb{S} \otimes \mathbb{C}[L]$, based on the rank n even integer lattice, L. The lattice L has the group of isometries, W. There is the simple way to define a linear action of W on V by $w : p \otimes e^x \mapsto w(p) \otimes e^{w(x)}$, but this will not be an automorphism of VOA structures in general. One has to make a modification and replace an action of W with an action of a group $\widetilde{W} \cong 2^n.W$. This construction applies to the case where L is the root lattice of a simple Lie algebra and gives the familiar subgroup of the torus normalizer lifting the Weyl group [Ti]. However, our argument is based only on properties of lattices and finite group theory, and uses no Lie theory. For this author, the ideas came from experience with centralizers of involutions in finite simple groups.

The tensor factor $\mathbb{C}[L]$ of V should be thought of as the quotient $\mathbb{C}[\hat{L}]/(e^1 + e^z)$ of the group algebra of $\hat{L}$ by the ideal generated by $e^1 + e^z$, which has the effect of making multiplication by e^z act as -1. Since $\widetilde{W}$ acts as automorphisms of $\hat{L}$, we get an action as automorphisms of its group algebra and the above quotient. The previous "naive" definition of the action of W on the space $\mathbb{C}[L]$ would not seem to (except in the degenerate case $(L, L) \leq 2\mathbb{Z}$) give algebra automorphisms of $\mathbb{C}[\hat{L}]/(e^1 + e^z)$ when this is identified with $\mathbb{C}[L]$ by a linear mapping of the form $e^x \mapsto c_x e^{\bar{x}}$, where bar indicates the natural map of $\hat{L}$ onto L, and $x \mapsto c_x$ is a set function to the nonzero complex numbers. Sometimes, one writes $V_L = \mathbb{S} \otimes \mathbb{C}[L]\tilde{}$ or $V_L = \mathbb{S} \otimes \mathbb{C}[L]_\epsilon$ to indicate that the second factor is identified linearly with the group algebra $\mathbb{C}[L]$ (the subscript ϵ refers to a cocycle giving $\hat{L}$ from L).

It follows that $\widetilde{W}$ has a natural action on the space V_L via the above natural action of W on the polynomial algebra $\mathbb{S}$.

Proposition 5.9 *$\widetilde{W}$ acts as automorphisms of the VOA V_L.*

Proof. Clearly, we have a degree preserving group of invertible linear transformations. It preserves the principal Virasoro element which has the form $\sum_i x_i(-1)^2$, for an orthonormal basis $\{x_i \mid i = 1, \ldots, rank(L)\}$, so corresponds to the natural quadratic form on the complex vector space spanned by the lattice. By linearity, the verification that $\widetilde{W}$ is a group of automorphisms is reduced to checking preservation of products of the form $a_n b$, where $a = p \otimes e^x$

and $b = q \otimes e^y$. For such elements, $Y(a,z)b$ has a clear general shape. At each z^{-n-1}, we get a finite sum of monomial expressions, each evaluated with the usual annihilation and creation operations and multiplication in the algebra $\mathbb{C}[\hat{L}]$. Since elements of $\widetilde{W}$ act as automorphisms of $\hat{L}$, it is easy to check that they preserve all the basic compositions involved in such monomials. $\square$

This completes our basic theory of $\widetilde{W}$.

See Def. 2.6 and Def. 2.4 for the definitions of W_X, D_X and T, which occur in the next result.

Proposition 5.10 *Let X be a lattice frame in L and let F be the associated VF, i.e., $\frac{1}{16}x(-1)^2 \pm \frac{1}{4}(e^x + e^{-x})$, $x \in X$.*

(i) In the group $\widetilde{W} \cong 2^n.W$, the stabilizer of F is just $2^n.D_X$.

(ii) In $N = N(T) = T\widetilde{W}$, the stabilizer of F has the form $N_F = (T \cap N_F)\widetilde{W}_X$, so as a group extension looks like $(T \cap N_F).W_X$.

Proof. (i) Obvious from the form of the action of $\widetilde{W}$.

(ii) Suppose $w \in \widetilde{W}$ and the coset Tw contains an element $g = tw$ in N_F. Such an element takes $\frac{1}{16}x(-1)^2 \pm \frac{1}{4}(e^x + e^{-x})$ to a vector in V_L of the form $\frac{1}{16}w(x)(-1)^2 \pm \frac{1}{4}(ae^{w(x)} + a^{-1}e^{-w(x)})$, for some nonzero scalar a. For this to be in F, we need $a = \pm 1$ and the image of w in W must be in W_X. $\square$

5.3 Nonsplit Extensions

We discuss extension theoretic aspects of a few frame stabilizers.

First take $k = 1$, for which $G \cong 2^{1+14}.Sym_{16}$. This group is a subgroup of $H := N_{Aut(V)}(G_{\mathcal{D}})$ and in the 16-dimensional orthogonal projective representation of H, it corresponds to the determinant 1 subgroup J_1 of a group $J \cong 2 \wr Sym_{16}$ stabilizing a double orthonormal basis D. We claim that J_1 does not contain a subgroup isomorphic to Sym_{16}, though it does obviously contain a subgroup A isomorphic to Alt_{16}. We may take A as the subgroup of J_1 stabilizing a orthonormal basis $B \subset D$.

We claim that J_1 does not contain a subgroup isomorphic to Sym_{16} or a central extension by a group of order 2. Suppose by way of contradiction that S is such a subgroup. The representation of J_1 is induced and the same is true for S. Let T be a relevant index 16 subgroup of S, stabilizing the 2-set $\{v, -v\}$ in D. Since the action of S is faithful, the induced representation for T must be faithful on $Z(T)$, which has order 1 or 2. If $Z(S)$ is nontrivial, $Z(S) = Z(T) = \{\pm 1\}$. It follows that T splits over $Z(T)$. This means that S splits over $Z(S)$ and so we may assume that $S \cong Sym_{16}$. So, the degree 16 representation for S is the standard degree 16 permutation module or that module tensored with

the degree 1 sign representation. Neither one gives a map of S to $SL(16,\mathbb{C})$, a contradiction which proves the claim.

Since $G/G_{\mathcal{D}} \cong J_1/Z(J_1) \cong 2^{14}{\cdot}Sym_{16}$, the claim implies that

$$1 \longrightarrow G_{\mathcal{C}}/G_{\mathcal{D}} \longrightarrow G/G_{\mathcal{D}} \longrightarrow Sym_{16} \longrightarrow 1$$

and

$$1 \longrightarrow G_{\mathcal{C}} \longrightarrow G \longrightarrow Sym_{16} \longrightarrow 1$$

are nonsplit. However, G does contain an extension $2{\cdot}Alt_{16}$.

For $k = 5$, G does not split over $G_{\mathcal{C}}$. If it did, the group denoted A in the proof of Th. 4.21 would be elementary abelian, a possibility which was disproved in that discussion.

5.4 Orbits of parabolic subgroups of orthogonal groups

Assume $\mathbb{F}$ is a perfect field of characteristic 2. We set up notation to discuss certain orbits of a parabolic subgroup in $O^+(2n,\mathbb{F})$.

Let W be a $2n$ dimensional vector space over the perfect field $\mathbb{F}$ of characteristic 2 and suppose that W has a nonsingular quadratic form Q with maximal Witt index, i.e., there are totally singular n-dimensional subspaces J, K so that $W = J \oplus K$ as vector spaces. Denote by $(\,.\,,\,.\,)$ the associated bilinear form: $Q(x+y) = Q(x) + Q(y) + (x,y)$, for all x, $y \in W$.

Let P be the subgroup of the isometry group $Aut(Q)$ which stabilizes J. Thus, P has the form $\mathbb{F}^{\binom{n}{2}}{:}GL(n,\mathbb{F})$ and is a maximal parabolic of $\Omega^+(2n,\mathbb{F})$.

Let Σ denote the set of n-dimensional subspaces which are totally singular with respect to the bilinear form but for which the set of singular vectors and zero forms a codimension 1 subspace. This is a nonempty set, for if $v \in W$ is nonsingular, $\mathbb{F}v + [J \cap v^\perp] \in \Sigma$. Since $\mathbb{F}$ is perfect, any two members of Σ are isometric.

Remark 5.11 We observe that if $A \in \Sigma$ and S_1, S_2 are any equal dimensional subspaces of S, the codimension 1 subspace in A consisting of zero and all singular vectors, then A/S_1 and A/S_2 are isometric because the isometry group $Aut(A) \cong AGL(S)$ induces $GL(S)$ on S.

Definition 5.12 For $A \in \Sigma$, we define the *J-indicator* $j(A) = dim(A \cap J)$. This is an integer from 0 to $n-1$, and all these values occur.

Proposition 5.13 *Two members of Σ are in the same P orbit if and only if they have the same J-indicator.*

Proof. One direction is obvious, so let us assume that A, $B \in \Sigma$ both have the same indicator d. The images of A, B in W/J are both $(n-d)$-dimensional,

so we may assume that the images are equal since P induces the full general linear group on W/J. It follows that $R := A \cap J = B \cap J$ is the radical of $J + A = J + B$.

Let A_1 complement R in A and B_1 complement R in B. Using the Remark, we know that A_1 and B_1 are isometric, say by an isometry ψ, which corresponds elements of A_1 and B_1 which are congruent modulo J: ψ is just the composite of isometries $A_1 \cong A/R \cong B/R \cong B_1$, where the middle isometry is based on congruence modulo J.

We now define an isomorphism of $J + A = J + B$ with itself by $\varphi : u + a \mapsto u + \psi(a)$, for $a \in A_1$. We verify that this is an isometry by using $a - \psi(a) \in J$: $Q(u + a) = Q(u) + Q(a) + (u, a) = Q(u) + Q(\psi(a)) + (u, \psi(a)) = Q(u + \psi(a))$, where Q is our quadratic form. Observe that this map takes $A = A_1 + R$ to $B = B_1 + R$.

Now, by Witt's theorem, φ extends to an isometry of W. Since it obviously fixes J, this extension lies in P. Therefore, A and B lie in a single P-orbit. $\square$

Proposition 5.14 *Let $A \in \Sigma$, let H its stabilizer inside P and let U be the unipotent radical of H (when $\mathbb{F}$ is a finite field of characteristic 2, $U = O_2(H)$). With $j = j(A)$, the J-indicator, one has $H/U \cong GL(j, \mathbb{F}) \times GL(n - j - 1, \mathbb{F})$.*

Proof. This is an exercise with actions of classical groups. We consider $0 \leq R := A \cap J \leq T := A + J = R^\perp \leq W$. Then R is the radical of T and T/R is a nonsingular space of maximal Witt index. Note that in T/R, J/R and A/R are maximal isotropic with respect to the bilinear form and give a direct sum decomposition of T/R as a vector space. Let J' be a complement in J to R and let A' be a complement in A to R. Then $M := J' + A'$ is a nonsingular subspace whose orthogonal complement contains R.

It follows that the action of H on M is the direct sums of the actions on J' and A', which are dual. The action could be as large as $GL(n - j, \mathbb{F})$ but is in fact just $AGL(n - j - 1, \mathbb{F})$ since A' is not totally singular with respect to the quadratic form. (More precisely, H stabilizes $S = \{x \in A \mid Q(x) = 0\}$ and is trivial on A/S.)

Note that the actions of H on R and W/R are dual, and are each $GL(j, \mathbb{F})$. This can be seen with action of a subgroup of H on $M^\perp$. $\square$

It is an exercise with linear algebra to work out the structure of U. For brevity, we record only the cases $j = 0$ and 4.

Corollary 5.15 *In the notation of the previous result, take $n = 5$ and $\mathbb{F} = \mathbb{F}_2$. Then $GL(4, 2)$ occurs as a quotient group of H for just $j = 0$ and $j = 4$. If $j = 0$, $|U| = 2^4$. If $j = 4$, $|U| = 2^{14}$.*

Proof. The previous result allows only $j = 0$ and $j = 4$. If $j = 0$, $W = J \oplus A$ implies that H acts faithfully on both factors. The case $j = 4$ is an exercise. $\square$

References

[A] A. V. Alekseevski, *Finite commutative Jordan subgroups of complex simple Lie groups,* Funct. Anal. Appl. **8** (1974), 277–279.

[B] A. Bruguières, *Catégories prémodulaires, modularisations et invariants des variétés de dimension* 3*,* Math. Annalen **316** (2000), 215–236.

[BRW] B. Bolt, T. G. Room and G. E. Wall, *On the Clifford collineation, transform and similarity groups. I, II.* J. Austral. Math. Soc. **2** (1961/1962), 60–79, 80–96.

[Ca] Roger Carter, *Simple groups of Lie type.* Reprint of the 1972 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1989.

[Ch] Claude Chevalley, *The Algebraic Theory of Spinors,* Columbia Univ. Press, Morningside Heights, 1954.

[CG] Arjeh Cohen and Robert L. Griess, Jr., *On simple subgroups of the complex Lie group of type* E_8*,* Proc. Symp. Pure Math. **47** (1987), 367–405.

[CS] John Conway and Neal Sloane, *Self dual codes over the integers modulo* 4*,* J. of Combinatorial Theory, Series A, **62** (1993), 30–45.

[DG] Chongying Dong and Robert Griess, *The automorphism group of a finitely generated VOA*, preprint.

[DGH] Chongying Dong, Robert Griess and Gerald Höhn, *Framed vertex operator algebras, codes and the moonshine module,* Comm. Math. Phys. **193** (1998), 407–448; q-alg/9707008.

[DGM] L. Dolan, P. Goddard and P. Montague, *Conformal field theory, triality and the Monster group,* Phys. Lett. B **236** (1990), 165–172.

[DGM2] L. Dolan, P. Goddard and P. Montague, *Conformal field theory of twisted vertex operators,* Nuclear Phys. B **338** (1990), 529–601.

[DN] Chongying Dong and Nagatomo, *Representations of the vertex operator algebra* V_L^+ *for a rank* 1 *lattice* L*,* to appear in Comm. Math. Physics.

[F] Walter Feit, *Characters of Finite Groups,* Benjamin, New York, 1967.

[FK] Igor Frenkel and Victor Kac, *Basic representations of affine Lie algebras and dual resonance models,* Invent. Math. **62** (1980), 23–66.

[FLM] Igor Frenkel, James Lepowsky and Arne Meruman, *Vertex Operator Algebras and the Monster,* Academic Press, Boston, 1988.

[FSS] J. Fuchs, A. N. Schellekens and C. Schweigert, *A matrix S for all simple current extensions,* Nucl. Phys. B **473** (1996), 323–366.

[Go] Daniel Gorenstein, *Finite Groups,* Harper and Row, New York, 1968; 2nd ed., Chelsea, New York, 1980.

[G73] Robert L. Griess, Jr. *Automorphisms of extraspecial groups and non-vanishing degree* 2 *cohomology,* Pacific Journal, **48**, 403–422, 1973.

[G76] Robert L. Griess, Jr., *On a subgroup of order* $2^{15}|GL(5,2)|$ *in* $E_8(\mathbb{C})$*, the Dempwolff group and* $Aut(D_8 \circ D_8 \circ D_8)$, J. Algebra **40** (1976) 271–279.

[G76b] Robert L. Griess, Jr., The structure of the "Monster" simple group, in: *Proceedings of the Conference on Finite Groups,* ed. W. Scott and F. Gross, Academic Press, New York, 1976, 113–118.

[G81] Robert L. Griess, Jr., *A construction of* F_1 *as automorphisms of a* 196883 *dimensional algebra,* Proc. Nat. Acad. USA, **78** (1981), 689–691.

[G82] Robert L. Griess, Jr., *The friendly giant,* Invent. Math. **69** (1982), 1–102.

[G86] Robert L. Griess, Jr., *The monster and its nonassociative algebra,* Contemporary Mathematics **45** (1985) 121–157, American Mathematical Society.

[G91] Robert L. Griess, Jr., *Elementary abelian p-subgroups of algebraic groups,* Geometriae Dedicata **39** (1991), 253–305.

[G96] Robert L. Griess, Jr., *Codes, Loops and p-locals,* in: Groups, Difference Sets and the Monster, ed. by K. T. Arasu, J. F. Dillon, K. Harada, S. Sehgal, R. Solomon; Walther de Gruyter, Berlin, New York, 1996.

[G98] Robert L. Griess, Jr., A vertex operator algebra related to E_8 with automorphism group $O^+(10,2)$, in: *The Monster and Lie Algebras,* ed. by J. Ferrar and K. Harada, De Gruyter, Berlin, 1998, 43–58.

[GR94] Robert L. Griess and A. Ryba, *Embeddings of* $U_3(8)$*,* $Sz(8)$ *and the Rudvalis group in algebraic groups of type* E_7*,* Inventiones Mathematicae **116** (1994), 215–241 (special issue dedicated to Armand Borel).

[GR99] Robert L. Griess and A. Ryba, *Finite simple groups which projectively embed in an exceptional Lie group are classified!,* Bulletin Amer. Math. Soc. **36** (1999), 75–93.

[H] Gerald Höhn, *Self-dual Vertex Operator Superalgebras with Shadows of large minimal weight*, Internat. Math. Res. Notices No. 13 **1997**, 613–621.

[H] Bertram Huppert, *Endliche Gruppen, I,* Springer Verlag, Berlin, 1967.

[I] I. M. Isaacs, *Character Theory of Finite Groups,* Academic Press, New York, 1976.

[J] D. G. James, *On Witt's Theorem for unimodular quadratic forms,* Pacific. J. Math. **26** (1968), 303–316.

[K] Victor G. Kac, *A remark on the Conway-Norton conjecture about the "Monster" simple group,* Proc. Nat. Acad. Sci. USA **77** (1980), 5048–5049.

[L] Ching H. Lam, *Code vertex operator algebras under coordinates change,* Comm. Algebra **27** (1999), 4587–4605.

[M96] Masahiko Miyamoto, *Griess algebras and conformal vectors in vertex operator algebras,* J. Algebra **179** (1996), 523–548.

[M98] Masahiko Miyamoto, *Representation theory of code vertex operator algebra,* J. Algebra **201** (1998), 115–150.

[M99] M. Müger, *Galois Theory for Braided Tensor Categories and the Modular Closure,* preprint 1999, math.CT/9812040 v3.

[MS] G. Moore and N. Seiberg, *Classical and quantum conformal field theory,* Comm. Math. Phys. **123** (1989), 177–254.

[N] V. V. Nikulin, *Integral Symmetric Bilinear Forms and their Applications,* Math. USSR Izvestija **14** (1980), 103–167.

[Sc] A. N. Schellekens *Meromorphic $c = 24$ Conformal Field Theories,* Comm. Math. Phys. **153** (1993), 159–185.

[Se] Graeme Segal, *Unitary representations of some infinite dimensional groups,* Comm. Math. Phys. **80** (1981), 301–342.

[Th] John Thompson, A simple subgroup of $E_8(3)$, in: *Finite Groups,* Iwahori, N. (ed.), Japan Society for the Promotion of Science, Tokyo, 1976, 113–116.

[Ti] Jacques Tits, *Sur les constantes de structure des algèbres de Lie semisimple complexes,* Inst. Hautes Études. Sci. Publ. Math, no. 31 (1966), 21–58.

[W] Helmut Wielandt, *Finite Permutation Groups,* Academic Press, New York, 1964.